UNOBSERVED MECHANISM DESIGN: EQUAL PRIORITY AUCTIONS

LI, HAO AND MICHAEL PETERS

VANCOUVER SCHOOL OF ECONOMICS
UNIVERSITY OF BRITISH COLUMBIA

MARCH 8, 2021

Abstract. We study the impact for mechanism design of the possibility that some participants are uninformed about the rules associated with a trading mechanism but otherwise rational. Since 'deviations' by the mechanism designer are not observed by these uninformed participants the nature of the 'equilibrium' of the design game changes, as do equilibrium mechanisms. We study the traditional independent private value auction environment and propose a method that makes it possible to characterize an interesting class of equilibrium outcomes for the game using standard reduced form direct mechanisms. We show that payoffs in the equilibrium where the seller's expected revenue is highest within this class can be characterized using a surprisingly simple mechanism called an *equal priority auction.* Informed bidders with intermediate valuations receive offers with the same probability as uninformed buyers, despite the fact the seller believes that the informed will accept the offers for sure, while uninformed buyers might not.

Li: hao.li@ubc.ca. Peters: peters@econ.ubc.ca. We are grateful to Social Sciences and Humanities Research Council of Canada for financial support. Gabriel Carroll, Vitor Farinha Luz, Alessandro Pavan, Sergei Severinov and Gabor Virag, provided helpful comments and suggestions.

1. Introduction

There is an acronym that floats around the internet - TLDNR - that explains why no one reads your email messages. It means "too long, didn't read". The long translation we adapt in this paper is "... there is undoubtedly information in your message, but its' value to me isn't likely to be as high as what I could get by reading something else". We refer to this as 'rational ignorance'.

The message of this paper is that this kind of behavior can impact trading mechanisms. We aren't the first to notice this. The marketing literature has documented buyers' tendency to ignore information when they make purchase decisions. The simplest commitment of all is a price commitment. Dickson and Sawyer (1990) asked buyers in supermarkets about their price knowledge as they were shopping. Only 50% of all respondents to their in store survey claimed to know the price of the object they had just taken off the supermarket shelf to put in their basket. Even when the item being placed in the basket had been specially marked down and heavily advertised, 25% of consumer did not even realize the good was on special.

Of course, having buyers be pleasantly surprised to learn that a price is lower than they expected isn't really a problem. The problem is the buyers who didn't know the price was on special, and went somewhere else to buy it. If prices can't influence buyer behavior, marketing has a problem.

We are interested in more than prices, we want to know how this kind of rational ignorance can impact trading mechanisms. We consider what is probably the best understood trading problem of all - the independent private value auction. We show that within a plausible class of equilibria - equilibria in which ignorant buyers convey no information to sellers - the revenue maximizing mechanism is something we call an *equal priority mechanism.*

The equal priority mechanism treats informed buyers and sellers with intermediate valuations in exactly the same way as ignorant buyers. When a mechanism chooses to attempt to trade with them, it makes them a take it or leave it price offer that is independent of any messages they may have sent. When buyers have very high or very low valuations, the seller treats messages as bids. If the seller decides to sell to one of these buyers, she will make an offer equal to the second highest bid she has received - much as she would in a standard auction.

Rationally ignorant buyers will be pooled with intermediate valuation buyers and receive a take it or leave it offer (which they might reject). In our formulation, this offer will be exactly the offer the buyers expected to receive. In other words, these buyers have rational expectations - there is nothing behavioral about them at all.

One appealing feature of the independent private value auction problem for mechanism design is that finding the revenue maximizing mechanism can be reduced to a problem of solving a maximization problem with a single parameter - the reserve price. The revenue maximizing mechanism with rationally ignorant buyers can be found by solving a problem with four parameters - harder, but still computationally tractable. The numerical solutions we have found in simple environments suggest that fixed priced trading is quite common. In fact, it is easy to show theoretically that if every buyer is equally likely to be informed or rationally ignorant, the trade will occur at a fixed price (with no auctions) much more than half the time. This may be another explanation for why auctions aren't particularly common in many consumer trading platforms.

One well known trading platform on which auctions *are* used is eBay. The environment on eBay doesn't fit our model exactly because buyers arrive randomly, but the mechanisms used by eBay resemble the equal priority action we describe below. A seller can implement something very close to what we describe here by running an auction with a 'buy it now' option, then offering the same model separately at a fixed price. Since buy it now options disappear on eBay once a buyer with a low valuation submits a bid, trade will occur at the fixed price (which should be the same as the buy it now price) a lot of the time, though auctions will continue to occur.

1.1. **Heuristic.** The formal derivation below is based on two arguments.

The first is that in an environment with uninformed buyers, standard auction mechanisms can't be supported as equilibrium even though the seller would much prefer to use them. The fault lies with the seller who can't resist the temptation of exploiting rationally ignorant buyers.

To see why, suppose the seller wants to use a second price auction with optimal reserve. This means that informed buyers read the auction rules, as they might on eBay, then realize they should bid their valuations. Uninformed buyers don't read the rules, so they only *anticipate* a second price auction. Acting on their expectations, they also bid their valuations.

What makes this break down is the fact that if the seller changes the auction rules, the uninformed won't realize it, and will continue to bid their valuations no matter what the seller does. A simple deviation can extract the surplus of the uninformed.

The seller can 'deviate' from the second price auction and ask for bidders to attach a coupon code to their bid. The coupon code isn't secret, it is plainly visible in the description of the bidding rules. A buyer who reads the new rules will see the coupon code and attach it to their bid.

A bidder who doesn't read won't add the code. The new mechanism commits to a second price auction for bids submitted with a code, but to treat bids with no code attached as if it were a first price auction. In other words, if the highest bid is submitted by an uninformed bidder, the seller will commit to make them an offer which is equal to their bid, instead of offering them the second highest bid.

The second argument involves how the seller should respond. The seller will want to sell to the uninformed buyers when informed buyers have low valuations. So the natural idea would be to have an auction, then if bids are too low, make an offer to the uninformed. The complication is that informed bidders don't have to bid. They can pretend to be uninformed. Since they are informed, they know when the seller will make an offer to the uninformed and what that offer will be. To prevent the informed from pretending to be uninformed, the seller has to keep the offer to the uninformed higher than she would like it to be, since the seller is never sure whether an uninformed buyer will accept the offer.

The seller then faces a trade off - keep the offer high and fully separate the informed from the uninformed, or lower the take it or leave it offer and allow some of the informed buyer to pool with the uninformed. We show that the latter is always revenue maximizing, which is where the equal priority phrase comes from in our title.

1.2. **Literature.** As mentioned above, the idea that consumers might not notice prices is an old one in the marketing literature, as in Dickson and Sawyer (1990) and references therein. The approach had been used earlier in economics, as in, say Butters (1977), in which buyers randomly observe price offers in a competitive environment. In that literature, firms advertise prices which some buyers see, while others do not.[1] These papers considered the same problem that we do, which is how this unobservability would affect the prices that firms offer. The difference here is that we are interested in mechanisms, not prices.

What ignorant buyers do is to provide type dependent outside options to informed buyers. This is one of the most basic problems in the literature on competing mechanisms. One example is the paper by McAfee (1993). His model had buyers whose outside option involved waiting until next period to purchase in a competing auction market just like the one in the current period. He imposed large market assumptions to ensure that the value of these outside options was independent of the reserve price that any seller in the existing market chose.

In our paper, the value of this outside option depends on the nature of the mechanism the seller chooses for the informed. This makes it resemble the later papers on competing mechanisms (at least in terms of outside options) like Virag (2010) who studies finite competing auction models where a seller who raises her reserve price increases congestion in other auctions, or Hendricks and Wiseman (2020) who study the same problem in a sequential auction environment.

With buyers potentially uninformed of the selling mechanism but nonetheless having rational expectations, the seller's commitment power is limited. There is an extensive literature on limited commitment (for example Bester and Strausz (2001), Kolotilin et al. (2013), Liu et al. (2014), or Skreta (2015)). To our knowledge, our model is the first to study commit with respect to a subset of traders involved in the same transaction.

A recent paper by Akbarpour and Li (2020) provides another model of limited commitment. They assume that each individual buyer only observes the part of the seller's commitment in relation to the buyer's own report, and impose a "credibility" constraint that the seller does not wish to secretly alter other parts of the commitment. The logic we described above explaining why the second price auction can't survive as an equilibrium is used in a similar way in their paper. The difference between their approach and ours is that they assume the credibility constraint applies to all buyers and describe mechanisms that are immune to this constraint. Here we assume that credibility is an issue only for some buyers and find optimal mechanisms.

Our informed buyers can 'prove' they are informed in the same sense as Ben-Porath et al. (2014). The main difference is that they assume that the social choice function is known by all the players, while in our model the driving force is the presence of buyers who are uninformed of the seller's mechanism. They also assume players have complete information about the state, but in our model only buyers know their own valuations.

Finally, our informed buyers can pretend they are uninformed but not the other way around. The one-sidedness of this incentive condition is similar to Denekere and Severinov (2006), who study an optimal non linear pricing problem with a

[1] See also Varian (1980), or Stahl (1994). Varian calls buyers informed if they see prices of all firms, and uninformed if they do not.

fraction of consumers constrained to reporting their valuations truthfully. As in our paper, a 'password' mechanism separates 'honest' consumers from 'strategic' consumers who can misrepresent their valuations costlessly. The main difference is that we start with a standard independent private value auction problem rather than a non linear pricing problem. More importantly, our uninformed buyers are uncommunicative in the class of equilibria we focus on, but they are rational rather than behavioral or face prohibitive communication cost.

2. Unobserved Mechanism Design

There are n potential buyers of a single homogeneous good. Each buyer has a privately known valuation w that is independently drawn from the interval $[0,1]$. Assume for the moment, all valuations are distributed according to some distribution F with strictly positive density f. Buyers' payoff when they buy at price p is given by $w-p$. The seller's cost is zero, so the profit from selling at price p is just p.

Each buyer is *informed* about the rules of the seller's mechanism with probability $(1-\alpha)$. Otherwise a buyer is rationally ignorant and pays no attention to the rules of the mechanism.

Define

$$\pi(w) = (1-F(w))w$$

as the revenue function from a take-it-or-leave-it offer w to uninformed buyers. In what follows,[2] we restrict attention to distribution functions such that $\pi(w)$ is strictly concave. Following the standard auction literature, we also define

$$\phi(w) = w - \frac{1-F(w)}{f(w)}$$

as the virtual valuation function for informed buyers. We have $\phi(0) < 0$ and $\phi(1) = 1$, and so $\phi(w)$ crosses 0 at least once. Since $\pi'(w) = -\phi(w)f(w)$, concavity of $\pi(\cdot)$ implies that $\phi(w)$ crosses 0 only once. Let the crossing point be r^*; this is also the unique maximizer of $\pi(w)$. Furthermore, $\phi(w)$ is strictly increasing in v for $w \geq r^*$.[3] The valuation r^* represents the optimal reserve price in a standard auction, regardless of the number of buyers.[4] That is, when $\alpha = 0$, the seller's outside option is always 0, so the reserve price is such that the virtual valuation of the buyer with w at the reserve price is equal to the seller's outside option.

There is a *common* message space $\mathcal{M}$ which is used by all buyers to communicate with the seller. For example, $\mathcal{M}$ might be the set of possible browsing histories for a buyer. A message from buyer i will be denoted b_i. We make no assumptions on

---

[2]The concavity assumption is used when we characterize the optimal equal priority auction and show that it achieves an equilibrium of our unobserved mechanism design game. It is not used in Theorem 1.

[3]At any $w \in (0,1)$, if $f(w)$ is non-decreasing, then by definition $\phi(w)$ is strictly increasing; if $f(w)$ is strictly decreasing at w and if $\phi(w) \geq 0$, then $\phi(w)$ is strictly increasing in w, because concavity of $\pi(w)$ implies that $\phi(w)f(w)$ is strictly increasing in w.

[4]In much of the auction literature, the seller has the fixed outside option of keeping the good. The virtual valuation function $\phi(w)$ is assumed to be strictly increasing to simplify the analysis (the "regular case" in Myerson (1981)). In our model, the seller's outside option in an auction with informed buyers is to give it to an uninformed buyer with a take-it-or-leave-it offer, and is endogenous. We do not need to assume that $\phi(w)$ is strictly increasing for valuations below r^*.

$\mathcal{M}$ itself except that it is rich enough to embed the product of the set of buyer valuations and the interval $[0,1]$. The message space $\mathcal{M}$ is common knowledge.

After processing all the buyers' messages, the seller's mechanism makes an *offer* to one of the buyers.[5] Our assumption is that it is common knowledge that all buyers understand and believe a take it or leave it price commitment.

This offer can be refused. For the moment, we'll assume that when it is, there is simply no trade at all. This makes for a cleaner analysis of the issues we are interested in. We'll explain later the sense in which the mechanism we describe will be part of an equilibrium even if multiple offers are allowed after the first one is rejected. We'll also defer until later to show the sense in which in this environment, sellers won't offer a mechanism that commits buyers to accept offers ex post.

Whether or not a buyer has taken the time to understand how a mechanism works is private information. In what follows we'll use the convention that a buyer who has figured out the mechanism is referred to as an informed buyer. One who hasn't is just called an uninformed buyer. So buyer i's type is given by the pair (v_i, τ_i), where $v_i \in [0,1]$ and $\tau_i \in \{\epsilon, \mu\}$, where ϵ means informed, and μ means uninformed. Each buyer has type μ with probability α that is independent of their valuation or the types of the other bidders.

A mechanism γ for the seller is a collection $\{\mathcal{M}, (p_i, q_i)_{i=1}^N\}$, where $\mathcal{M}$ is the common message space, p_i is a mapping from a profile of messages $(b_1, \ldots, b_N)$ to a take-it-or-leave-it offer designed for buyer i, while q_i maps the same set of messages into the probability with which the offer will actually be made to buyer i. The mapping q_i must satisfy

$$\sum_i q_i(b_1, \ldots, b_N) \leq 1$$

for every profile of messages. Let Γ be the set of feasible mechanisms.[6]

Definition. The imperfect information game $\mathcal{G}(\alpha)$ is defined to be the extensive form game of imperfect information in which the seller first commits to some $\gamma \in \Gamma$, the informed buyers send messages to the seller that depend on γ, and the uninformed send messages that are independent of γ. Allocations and final payoffs are determined by the mechanism γ, the realized messages, and the acceptance decisions of any buyer who receives an offer. The parameter α gives the common belief with which each buyer and the seller believes that each of the buyers is uninformed.

A strategy rule in $\mathcal{G}(\alpha)$ for buyer i is a function $\sigma_i : [0,1] \times \{\epsilon, \mu\} \times \Gamma \to \triangle(\mathcal{M})$ that specifies what message the buyer will send for each of the valuations conditional on whatever the buyer knows about the seller's mechanism. Since an

[5]In the standard mechanism design paradigm, a mechanism produces an allocation according to a mapping from messages. Representing the output of a mechanism as an offer instead of an allocation has no implications. This is no longer the case in our unobserved mechanism problem. See section 4.1 for additional comments on modeling the output of an unobserved mechanism as an algorithm.

[6]We have thus restricted each mechanism γ in Γ to be a "pure" one, in that each p_i maps a profile of messages to a single price, instead of a distribution of prices. This can be easily dropped without affecting the formalism; for the equilibria we construct in the paper this restriction is without loss.

uninformed buyer never sees the mechanism a seller offers, we have the informational constraint

$$\sigma_i(v_i,\mu,\gamma)=\sigma_i(v_i,\mu,\gamma')=\sigma_i(v_i,\mu)$$

for all γ and γ'. We retain this assumption throughout the paper.

As mentioned above, informed buyers can pretend to be uninformed, but not conversely. This allows the seller to identify informed buyers by, for example, providing a coupon code that must be submitted with a bid. One of the stranger properties of equilibrium in this game is that pure strategy (for the seller) equilibrium typically won't exist. If such equilibrium did exist, uninformed bidders would guess the coupon code. This is just because of the fact that strategies are common knowledge in any Nash based equilibrium. This is just another way of saying that uninformed bidders have 'rational expectations'.

Once the uninformed guess the coupon code, they will submit informative bids and the seller won't be able to prevent himself from exploiting that information. To prevent the uninformed buyers from guessing this password, it has to be random. There is nothing secret about this password, it is freely available to anyone who takes the time to read the rules of the mechanism.

Refer to the seller's mixture as $\psi\in\triangle(\Gamma)$. Let $R\left(\gamma,(\sigma_i(\cdot,\cdot,\gamma))_{i=1}^n\right)$ be the expected revenue for the seller from mechanism γ when an uninformed buyer i uses strategy $\sigma_i(\cdot,\mu)$ an informed buyer i uses $\sigma_i(\cdot,\epsilon,\gamma)$.

A perfect Bayesian equilibrium for this game is a mixture ψ for the seller, and pairs of strategy rules $\left(\sigma_i(\cdot,\epsilon,\gamma),\sigma_i(\cdot,\mu)\right)_{i=1}^N$ for informed and uninformed buyers respectively that satisfy the usual conditions:[7] for each informed buyer i,

(2.1)
$$\mathbb{E}_{v_{-i},\tau_{-i}}\left\{q_i\left(\sigma_i(v_i,\epsilon,\gamma),\sigma_{-i}(v_{-i},\tau_{-i},\gamma)\right)\cdot\max\left[v_i-p_i\left(\sigma_i(v_i,\epsilon,\gamma),\sigma_{-i}(v_{-i},\tau_{-i},\gamma)\right),0\right]\right\}$$

(2.2)
$$\geq\mathbb{E}_{v_{-i},\tau_{-i}}\left\{q_i\left(b',\sigma_{-i}(v_{-i},\tau_{-i},\gamma)\right)\cdot\max\left[v_i-p_i\left(b',\sigma_{-i}(v_{-i},\tau_{-i},\gamma)\right),0\right]\right\}$$

for all $v_i\in[0,1]$, $b'\in\mathcal{M}$ and realized γ from the mixture ψ; for each uninformed buyer i,

(2.3)
$$\mathbb{E}_{v_{-i},\tau_{-i},\gamma}\left\{q_i\left(\sigma_i(v_i,\mu),\sigma_{-i}(v_{-i},\tau_{-i},\gamma)\right)\cdot\max\left[v_i-p_i\left(\sigma_i(v_i,\mu),\sigma_{-i}(v_{-i},\tau_{-i},\gamma)\right),0\right]\right\}$$

(2.4)
$$\geq\mathbb{E}_{v_{-i},\tau_{-i},\gamma}\left\{q_i\left(b',\sigma_{-i}(v_{-i},\tau_{-i},\gamma)\right)\cdot\max\left[v_i-p_i\left(b',\sigma_{-i}(v_{-i},\tau_{-i},\gamma)\right),0\right]\right\}$$

for all $v_i\in[0,1]$, and $b_i\in\mathcal{M}$; and for the seller,

$$\mathbb{E}_\gamma\left\{R\left(\gamma,(\sigma_i(\cdot,\cdot,\gamma))_{i=1}^n\right)\right\}\geq R\left(\gamma',(\sigma_i(\cdot,\cdot,\gamma'))_{i=1}^n\right)\qquad(2.5)$$

for all $\gamma'\in\Gamma$.

2.1. **Relationship to standard mechanism design.** Our unobserved mechanism design game is a kind of informed principal problem. The unusual part about it is that the seller has more information about the mechanism he is using than about the product itself. The equilibrium we describe has a kind of 'punishment by beliefs' aspect that is based on this.

---

[7]For simplicity we assume below that each buyer uses a pure strategy. It is straightforward to extend the following conditions to allow for mixing by buyers. The max operation appears when taking expectations because a mechanism generates an offer instead of an outcome.

For example, we have restricted the game to one in which the seller makes just a single take it or leave offer after processing messages. The seller might like to make additional offers if he makes an offer to the uninformed which is rejected. However, if the uninformed believe there is only a single offer, they can support the equilibrium by simply disappearing if they don't get the first offer. If the seller understands this he won't find it profitable to alter his mechanism to introduce these second offers.

Like all Bayesian games, in our unobserved mechanism design game, equilibrium strategies must constitute a fixed point. In its simplest form, uninformed buyers' expectations about the relationship between the messages they send and the offers they get must coincide with the actual relationship once the seller best replies to those expectations. As a result, our game potentially has many equilibrium outcomes. The source of multiplicity comes from different information contents of messages by uninformed buyers.

A babbling equilibrium is one example. No uninformed buyer will send an informative message because he or she believes the seller's mechanism won't respond to it. Since the seller thinks uninformed buyers are babbling, there is no reason for their mechanism to respond to these messages.

There may exist equilibrium outcomes in which uninformed buyers with very low valuations will separate from higher valuation buyers by saying they aren't interested in an offer. To support this, some of the uninformed who believe they will never accept a seller offer must nonetheless act as if they might accept an offer. Messages from uninformed buyers are informative of their valuations, but they expect at most one 'serious' offer. These outcomes can be constructed as straightforward extensions of what we describe below,[8] but the challenge is to show that the seller will commit not to make offers to uninformed buyers who say they are uninterested for incentive reasons (even when he has no better alternative).

There may also exist equilibria in which uninformed buyers send much richer information with their messages, which are effectively cheap talk. All of these alternative equilibria where uninformed buyers convey information to the seller exhibit the same logic that we describe below. For example, informative cheap talk messages allow the seller to make offers to the buyers whose messages say they are most likely to accept them. This effect tends to raise the seller's expected revenues. Yet to maintain incentive compatibility the seller has to make fairly attractive offers so that the informed are incentivized to reveal that they are informed. This makes it hard to determine whether equilibria with more information conveyed by uninformed buyers actually benefit the seller.

A natural approach to find an equilibrium outcome as a fixed point is to use the revelation principle and all the well known properties of reduced form mechanisms. The usual composition of the outcome function and the strategies can be used to create something that looks like a direct mechanism. Yet not all direct mechanisms that look incentive compatible correspond to equilibrium outcomes because

[8]To be precise, for any threshold valuation sufficiently close to 0, we can construct an equal priority auction in which informed buyers with intermediate valuations are pooled together with uninformed buyers with valuations above the threshold (who tell the seller that they are interested in an offer). Further, the equal priority auction is revenue maximizing conditional on the seller not making an offer to uninformed buyers with valuations below the threshold.

uninformed buyers can only use a restricted set of communication strategies. In addition, equilibrium outcomes in which uninformed buyers learn how to participate in a direct mechanism can't correspond to equilibrium outcomes.

A conceptually workable approach is to condition the search for equilibrium outcomes as fixed points on a particular communication strategy of uninformed buyers. Given this strategy, the seller solves an optimal mechanism design problem where the objective function includes both revenues from uninformed and informed buyers, but incentive constraints are imposed only on informed buyers. These constraints require informed buyers to report their valuations truthfully and not to have strict incentives to pretend to be uninformed by adopting the latter's communication strategy. There are no incentive constraints on uninformed buyers at this stage, even if the communication strategy is informative of their valuations and is exploited by the seller. If an optimal mechanism of this kind can be characterized, then one may hope to find a fixed point by showing that the communication strategy of uninformed buyers is indeed a best response to the optimal mechanism. This last step relies on randomization by the seller in implementing the optimal mechanism to prevent uninformed buyers from participating in the mechanism in the same way as informed buyers.

There are at least three difficulties with the above approach. We have to be able to characterize optimal mechanisms for a given communication strategy of uninformed buyers. Finding a fixed point in terms of a particular communication strategies within a class can be hard. It is not clear what class of communication strategies should be considered.

2.2. **Direct mechanisms.** So we are going to define a special kind of direct mechanism that we can use to characterize an important class of equilibrium outcomes. In particular, it will allow us to characterize equilibrium in which the seller uses a symmetric mechanism, informed buyers use the same strategy, and uninformed buyers use uninformative messages (babble).

First we introduce notation to help define what *symmetry* means. In what follows the notation m always means the number of uninformed buyers. We reorder n buyers such that the first $n-m$ of them are informed; the orders among the informed and among the uninformed are arbitrary. For each $v = (v_1, \ldots, v_n) \in [0,1]^n$, and for each $i = 1, \ldots, n-m$, let

$$\rho_m^i(v) = (v_i, v_2, \ldots, v_{i-1}, v_1, v_{i+1}, \ldots, v_{n-m}, v_{n-m+1}, \ldots, v_n\};$$

that is, $\rho_m^i(v)$ switches the positions of v_1 and v_i. Now we have

Definition. A direct mechanism δ is a collection of functions

$$\left\{ (q_m^\epsilon, p_m^\epsilon)_{m=0}^{n-1}, (q_m^\mu, p_m^\mu)_{m=1}^{n} \right\}$$

where $q_m^\tau, p_m^\tau : [0,1]^n \to [0,1]$, $\tau = \epsilon, \mu$, satisfy

- $(q_m^\tau(v), p_m^\tau(v))$, $\tau = \epsilon, \mu$, are invariant to $(v_{n-m+1}, \ldots, v_n)$;
- $(q_m^\epsilon, p_m^\epsilon)$ are invariant to permutations of $(v_2, \ldots, v_{n-m})$, and (q_m^μ, p_m^μ) are invariant to permutations of $(v_1, \ldots, v_{n-m})$;
- for all v and for all m,

$$\sum_{i=1}^{n-m} q_m^\epsilon \left(\rho_m^i (v) \right) + m q_m^\mu(v) \le 1. \tag{2.6}$$

The function $q_m^{\mu}(v)$ gives the probability with which an offer $p_m^{\mu}(v)$ is made to an uninformed buyer given that there are m uninformed buyers and the profile of valuations is $v = \{v_1, \ldots, v_n\}$. The function $q_m^{\epsilon}(v)$ gives the probability with which an offer $p_m^{\epsilon}(v)$ is made to buyer 1 given that there are m uninformed buyers and the valuation profile of buyers $i = 2, \ldots, n$ is $v_{-1} = \{v_2, \ldots, v_n\}$. Since uninformed buyers babble, we require the allocation and the offer functions of both the informed and the uninformed to be independent of the valuations of the latter. Symmetry requires the allocation and the offer functions of uninformed buyers to be invariant to permutations of the valuation profile of the informed, and the allocation and the offer functions of each informed buyer to be invariant to permutations of the valuation profile of the other informed buyers. Since $\rho_m^i(v)$ switches the positions of the first element of v and its i-th element, the sum $\sum_{i=1}^{n-m} q_m^{\epsilon}\left(\rho_m^i(v)\right)$ gives the probability that the offer is made to one of the first $n-m$ elements of v. Then (2.6) ensures that when the informed buyers have valuations given by the first $n-m$ valuations in v, the probability with which the good is offered to one of them plus the probability that it is offered to one of the uninformed buyers is less than or equal to 1.

We can use the above definitions to build something that looks exactly like a traditional reduced form mechanism. The probability with which an informed buyer whose valuation is w receives an offer when there are m uninformed is

$$Q_m^{\epsilon}(w) = \mathbb{E}_v\left\{q_m^{\epsilon}(v) | v_1 = w\right\}.$$

Similarly

$$P_m^{\epsilon}(w) = \mathbb{E}_v\left\{q_m^{\epsilon}(v) p_m^{\epsilon}(v) | v_1 = w\right\}$$

is the expected price the informed bidder with valuation w would pay. Note that we have assumed that in any direct mechanism an informed buyer accepts the offer he receives with probability one. The is no max operator for informed buyers. This assumption is justified because informed buyers know the mechanism.

For each $m = 0, \ldots, n-1$, let $B(m; n-1, \alpha)$ be the probability that there are m uninformed buyers among the $n-1$ others. This probability is given by

$$B(m; n-1, \alpha) = \binom{n-1}{m}(1-\alpha)^{n-1-m}\alpha^m.$$

Now by taking expectations over m we have the usual reduced form functions:

$$Q^{\epsilon}(w) = \sum_{m=0}^{n-1} B(m; n-1, \alpha) Q_m^{\epsilon}(w),$$

$$P^{\epsilon}(w) = \sum_{m=0}^{n-1} B(m; n-1, \alpha) P_m^{\epsilon}(w).$$

We then have

$$U^{\epsilon}(w) = wQ^{\epsilon}(w) - P^{\epsilon}(w).$$

At this point, we inherit all the usual results from mechanism design in iid environments for each of the informed buyers. In particular, if the mechanism δ is incentive compatible *with respect to valuations*, the payoff to an informed buyer with valuation w can be written as

$$U^{\epsilon}(w) = \int_0^w Q^{\epsilon}(x)\, dx, \tag{2.7}$$

with $Q^{\epsilon}(\cdot)$ non-decreasing.[9] The (interim) payoff to an uninformed bidder with valuation w is

$$U^{\mu}(w) = \sum_{m=0}^{n-1} B(m; n-1, \alpha)\mathbb{E}_v \left\{ q^{\mu}_{m+1}(v) \max\left[w - p^{\mu}_{m+1}(v), 0\right] \right\}.$$

Definition. The mechanism δ is incentive compatible for informed buyers if (2.7) holds, $Q^{\epsilon}(\cdot)$ is non-decreasing and

$$U^{\epsilon}(w) \geq U^{\mu}(w)$$

for every w.

From standard arguments and properties of the binomial distribution, it is straightforward to show that the seller's revenue from informed buyers (again from any incentive compatible mechanism) is given by

$$n(1-\alpha)\int_0^1 Q^{\epsilon}(w)\,\phi(w) f(w) dw.$$

The seller's revenue from uninformed buyers is given by

$$\sum_{m=1}^{n} B(m; n, \alpha)\mathbb{E}_v \left\{ m q^{\mu}_m(v) \pi\left(p^{\mu}_m(v)\right) \right\}.$$

The seller's total revenue $R(\delta)$ from a direct mechanism δ is the sum of the above two expressions.

The following result provides a two-way relationship between the optimal direct mechanism and a symmetric equilibrium of the unobserved mechanism design game with babbling by uninformed buyers. Any equilibrium outcome can be found by characterizing optimal direct mechanisms, and an optimal direct mechanism can be used to construct an equilibrium.

Theorem 1. *Suppose that uninformed buyers send uninformative messages in some symmetric equilibrium of the game $\mathcal{G}(\alpha)$. Then there is an incentive compatible direct mechanism δ^* that achieves the equilibrium expected revenue for the seller and $R(\delta^*) \geq R(\delta)$ for every incentive compatible direct mechanism δ. Conversely, any incentive compatible direct mechanism δ^* that maximizes $R(\delta)$ can be used to construct an equilibrium in the Bayesian game $\mathcal{G}(\alpha)$.*

Our argument for going from an equilibrium outcome to an incentive compatible direct mechanism follows the standard revelation principle. The assumption that the equilibrium has uninformative messages from buyers is used because direct mechanisms as we have defined them do not allow an allocation to depend on the valuations of the uninformed. In a direct mechanism, the equilibrium condition (2.2) for informed buyers becomes incentive compatibility condition for truthful bidding and a participation condition with type dependent outside option from pretending to be uninformed. The equilibrium condition (2.5) for the seller requires the incentive compatible direct mechanism derived from the equilibrium to be revenue maximizing.

[9] See, for example, Myerson (1981). We have assumed $U^{\epsilon}(0) = 0$ for simplicity. This is usually not part of requirement for incentive compatibility, but clearly necessary for any revenue maximizing direct mechanism.

In the other direction, from an optimal direct mechanism, we construct a symmetric equilibrium of the unobserved mechanism design game $\mathcal{G}(\alpha)$ where uninformed buyers babble. This is done through introducing a random password in the direct mechanism. Informed buyers are identified as those who match the realized password, and their valuation reports are accepted as truthful, while valuation reports from buyers who don't match the password have their valuation reports ignored by the seller. In equilibrium then uninformed buyers babble, and the seller finds it optimal to commit to mechanisms that ignore buyers who can't match the realized password.

3. Equal-priority auctions

Our main result is that for distributions such that $\pi(\cdot)$ is concave, the outcome of a symmetric equilibrium of the game $\mathcal{G}(\alpha)$ where uninformed buyers babble corresponds to a revenue maximizing "equal priority auction." We'll show this in two parts. First we'll describe the set of equal priority auctions and describe one that gives the seller the highest expected revenue. Later we'll show how to verify this is the best for the seller among all direct mechanisms.

An equal priority auction is fully characterized by four numbers, a 'reserve price' r, a price offer t, and the upper and lower bound v_+ and v_- of an interval of buyer types. We'll assume throughout that $r \leq t \leq v_- \leq v_+$.

In what follows, there is some message that is treated as if the buyer who sent that message is uninformed. Each of the informed buyers sends a bid. A realized profile of messages and bids will then have m messages saying uninformed, and $n-m$ bids. Denote the number of informed buyers who bid in the interval $[v_-, v_+]$ as k. The auction treats the k informed buyers and m uninformed buyers with the same allocation priority. Priorities of informed buyers who bid above v_+ and who bid below v_- are equal to their bids, with the former all higher than the $(m+k)$ buyers and the latter all lower than them. The allocation and offers in an equal priority auction are determined in the following way:

- If $m \geq 1$ and the highest bid received from the informed bidders is no larger than v_+, then the seller makes an offer t to each uninformed bidder and an offer v_- to each informed bidder who bid in the interval $[v_-, v_+]$ with probability $1/(m+k)$.
- Otherwise, the seller makes an offer to the informed buyer who made the highest bid. Let v' be the second highest bid by an informed buyer. The offer to the high bidder is

$$\begin{cases} v' & v' > v_+ \\ r & m = 0; v' < r \\ v' & m = 0; v' \in (r, v_-) \\ \frac{v_- + (m+k)v_+}{m+k+1} & \text{otherwise.} \end{cases}$$

These rules constitute an indirect mechanism and support some kind of Bayesian equilibrium in bidding strategies. Our main theorem is going to say that conditional on uninformative messages from the uninformed bidders, the revenue maximizing mechanism is going to be a special kind equal priority auction. To see what that means, and to understand how to find the optimal one, one bit of notation is required. Suppose for the moment, potentially counterfactually, that informed buyers bid their true valuations Then using the allocation rule in the indirect mechanism, we can calculate the probability with which each type of informed buyer trades. This probability of trade function Q^{ϵ} for an informed buyer is given by

$$\begin{cases} 0 & \text{if} \quad w < r \\ (1-\alpha)^{n-1}F^{n-1}(w) & \text{if} \quad w \in [r, v_-) \\ \sum_{m=0}^{n-1} B(m; n-1, \alpha) \sum_{k=0}^{n-1-m} B_k^{n-1-m}(v_-, v_+)/(m+k+1) & \text{if} \quad w \in [v_-, v_+] \\ \sum_{m=0}^{n-1} B(m; n-1, \alpha) F^{n-1-m}(w) & \text{if} \quad w > v_+, \end{cases} \tag{3.1}$$

where

$$B_k^{n-1-m}(v_-, v_+) = \binom{n-1-m}{k} (F(v_+) - F(v_-))^k F^{n-1-m-k}(v_-).$$

For informed buyers with valuation w between r and v_-, $Q^{\epsilon}(w)$ is such that trade occurs only when there are no uninformed buyers who have a higher priority. For $w > v_+$, we have

$$Q^{\epsilon}(w) = ((1-\alpha) F(w) + \alpha)^{n-1},$$

so informed buyers with valuation w above v_+ have a higher priority than uninformed buyers.

For convenience, we denote $Q^{\epsilon}(w)$ for $w \in [v_-, v_+]$ as $\chi(v_-, v_+)$. To provide a convenient formula, we re-do the double summations over m and k by first summing over k for fixed $l = m + k$, and then summing over l. We can rewrite $\chi(v_-, v_+)$ as

$$\sum_{l=0}^{n-1} \binom{n-1}{l} ((1-\alpha)F(v_-))^{n-1-l} \frac{1}{l+1} \sum_{k=0}^{l} \binom{l}{k} ((1-\alpha)(F(v_+) - F(v_-)))^k \alpha^{l-k}$$
$$= \sum_{l=0}^{n-1} \binom{n-1}{l} ((1-\alpha)F(v_-))^{n-1-l} \frac{1}{l+1} ((1-\alpha)(F(v_+) - F(v_-)) + \alpha)^l.$$

Thus,

$$\chi(v_-, v_+) = \frac{((1-\alpha)F(v_+) + \alpha)^n - ((1-\alpha)F(v_-))^n}{n((1-\alpha)(F(v_+) - F(v_-)) + \alpha)}. \tag{3.2}$$

The function χ gives the probability that a buyer whose valuation is in the pooling interval $[v_-, v_+]$ receives an offer. The logic in $\chi(v_-, v_+)$ is that an informed bidder has the same chance of receiving an offer as any of the uninformed buyers and informed buyers whose valuations are in the interval $[v_-, v_+]$ as long as none of the other informed bidders has valuation above v_+. This explains why in the formula (3.2) the denominator is the expected number of buyers who have the equal priority, and the numerator is the total probability that there is one with the priority.

The trading probability $Q^{\epsilon}(w)$ of an informed buyer with valuation w is weakly increasing. It is continuous except at three valuations. It jumps up at $w = r$ from 0 to $Q^{\epsilon}(r)$. Another upward jump occurs at $w = v_-$:

$$\chi(v_-, v_+) > B(0; n-1, \alpha) B_0^{n-1}(v_-, v_+) = (1-\alpha)^{n-1} F^{n-1}(v_-).$$

It jumps up for the third time at $w = v_+$:

$$\chi(v_-, v_+) < \sum_{m=0}^{n-1} B(m; n-1, \alpha) \sum_{k=0}^{n-1-m} B_k^{n-1-m}(v_-, v_+) = ((1-\alpha)F(v_+) + \alpha)^{n-1}.$$

Then mimicking direct mechanisms we could define an expected payoff $U^{\epsilon}(w)$ to an informed buyer as follows:

$$U^{\epsilon}(w) = \int_0^w Q^{\epsilon}(x)dx. \tag{3.3}$$

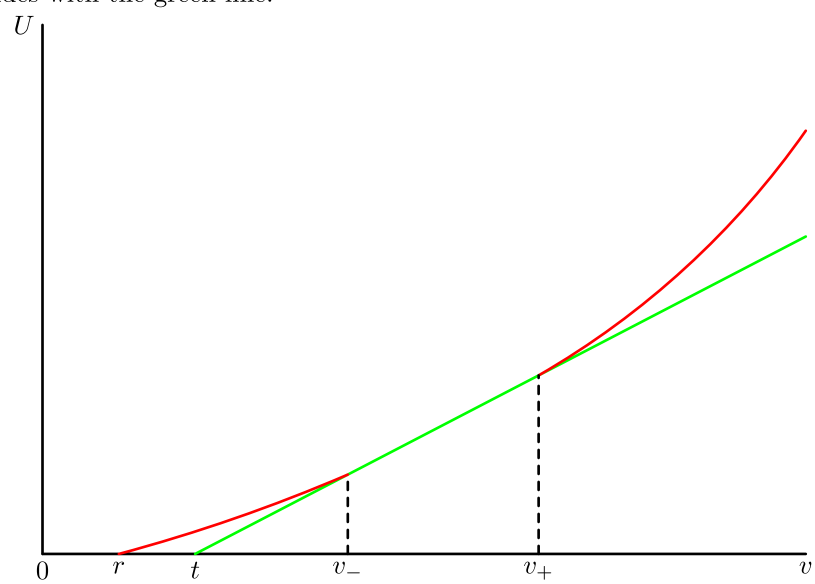
We have the following result.

Lemma 2. *There is a Bayesian equilibrium in truthful bidding strategies if*

$$\int_r^{v_-} (1-\alpha)^{n-1} F^{n-1}(w)dw \geq \chi(v_-, v_+)(v_- - t) \tag{3.4}$$

Two arguments are needed. The first is to show that the transfers defined above together with the allocation rule are the ones that make truthful bidding incentive compatible by informed buyers. Note when informed buyers bid their valuations truthfully, they accept their offers with probability one. Since the allocation rule is monotone, we accomplish the first stope by showing that the payoff of informed buyers from truthful bidding matches the payoff defined by (3.3) and (3.1) (Myerson,1981). The second is to show that when t satisfies condition (3.4) no informed buyer can improve his or her payoff by pretending to be uninformed. This just follows from the observation that the right hand side of (3.4) is the expected payoff for an informed buyer with valuation v_- pretending to be uninformed. By construction, uninformed buyers have the same allocation priority as informed buyers whose valuations are in $[v_-, v_+]$. The expected payoff of informed buyers given by (3.3) and (3.1) is strictly convex between r and v_- and above v_+. Thus, the incentive condition for informed buyers not to pretend to be uninformed is satisfied if and only if it holds for an informed buyer with valuation v_-.

The following figure shows the Bayesian equilibrium payoffs to bidders with various valuations in an equal priority auction with a binding incentive compatible constraint (3.4). The green line represents the payoff each buyer type achieves by acting as an uninformed bidder. The red curve represents the payoff to informed bidders - except that the payoff to informed bidders who bid in the interval $[v_-, v_+]$ coincides with the green line.



In an equal priority auction with a binding incentive compatible constraint (3.4), it is a matter of indifference for informed buyers with valuations in $[v_-, v_+]$ whether they participate in the auction by truthfully reporting their valuations, or wait for the take-it-or-leave-it offer t just like an uninformed buyer. Indeed, the same truth telling equilibrium among informed buyers is implemented if we change the transfer rule, so that an informed buyer with valuations in the pooling interval $[v_-, v_+]$ receives the offer t, instead of the maximum of the second highest bid and reserve price r when there are no other buyers in the equal priority pool, or v_- when there is at least one buyer in the pool. Informed buyers with low valuations, between r and v_-, and those with high valuations, above v_+, have strict incentives to participate in the auction.

3.1. **Revenue Maximizing Equal Priority Auction.** This already looks like a direct mechanism, albeit one with very specific allocation rules. The seller's expected revenue from informed buyers is given by

$$n(1-\alpha)\int_r^1 Q^\epsilon(w)\phi(w)f(w)dw, \tag{3.5}$$

and the revenue from uninformed buyers is given by

$$\sum_{m=1}^{n} B(m;n,\alpha)\sum_{k=0}^{n-m} B_k^{n-m}(v_-,v_+)\frac{m}{m+k}\pi(t) = n\alpha\chi(v_-,v_+)\pi(t). \tag{3.6}$$

The revenue maximizing equal-priority auction $\{r, t, v_-, v_+\}$ maximizes the sum of (3.5) and (3.6) subject to

$$r \le t \le v_- \le v_+;$$

and (3.4). The following lemma characterizes optimal equal-priority auctions.

Lemma 3. *If (r, t, v_-, v_+) is an optimal equal-priority auction, then*

$$0 < r < r^* < t < v_- < v_+ < 1.$$

Further, (3.4) holds with equality, and

(3.7)

$$\alpha(\pi(t)-\phi(v_+)) = (1-\alpha)\left((v_- - t)(\phi(v_+)-\phi(v_-))f(v_-) + \int_{v_-}^{v_+} f(w)(\phi(v_+)-\phi(w))dw\right);$$

(3.8)

$$-\alpha\pi'(t) = (1-\alpha)(\phi(v_+)-\phi(v_-))f(v_-);$$

(3.9)

$$-\phi(r)f(r) = (\phi(v_+)-\phi(v_-))f(v_-).$$

The three conditions (3.7), (3.8) and (3.9) are just the first order conditions for an interior optimum. To establish that the optimal auction is indeed interior, satisfying $0 < r < t < v_- < v_+ < 1$, our proof (in the appendix) uses a variational argument.

In a revenue maximizing equal priority auction, the reserve price r for selling to informed buyers with low valuations (below v_-) is set below the standard optimal reserve price r^* in the absence of uninformed buyers, as can be seen from (3.9). This sacrifices revenue when all informed buyers have low valuations and there are no uninformed buyers, but provides incentives for informed buyers whose valuations are low but close to v_- to participate in the auction instead of pretending to be uninformed. Correspondingly, (3.8) implies that the take-it-or-leave-it price t to uninformed buyers is raised above the optimal monopoly price r^* in the absence of informed buyers. This reduces the revenue when all buyers are uninformed, but provides disincentive for informed buyers to pretend to be uninformed.

If the seller does not give the good to an informed buyer, he can always make a take-it-or-leave-it offer to an uninformed buyer if there is one. Absent of incentives, the seller would set the reserve price $\overline{r}(t)$ for informed buyers so that the virtual valuation is equal to the expected profit $\pi(t)$ of making the offer t to an uninformed buyer:

$$\phi(\overline{r}(t)) = \pi(t).$$

However, by condition (3.7), the optimal equal priority auction has $\phi(v_+) < \pi(t)$. This means that the seller gives the good to informed buyers even though their virtual valuations are lower than the value of the seller's 'outside option' $\pi(t)$. This reason for doing this is to provide incentives for informed buyers with valuations just above v_+ to participate in the auction rather than wait for the take-it-or-leave-it offer.

The interval $[v_-, v_+]$ is non-degenerate as long as uninformed buyers are present in the model, i.e., $\alpha > 0$. Briefly if the interval is degenerate, the seller can raise expected revenue by cutting the price t that he offers to the uninformed. The downside is that he loses revenue from the informed who are pooled together with the uninformed. A variational argument can be used more generally to show that the cutting the price offer to the uninformed has a first order impact on profits, while the loss from the informed is second order.

When all bidders are surely informed the revenue from the optimal equal priority auction converges to the revenue from the standard auction with reserve price r^*, as it becomes optimal for the seller not to distort the reserve price r at all to provide incentives (equation 3.9). The pooling interval shrinks to a single valuation v_0 as α goes to 0, satisfying the binding constraint (3.4) that an informed buyer with valuation v_0 is indifferent between participating in the auction and receiving a take-it-or-leave-it offer t_0 when all other buyers have valuations below v_0,[10]

$$\int_{r^*}^{v_0} F^{n-1}(w)dw = F^{n-1}(v_0)(v_0 - t).$$

The limit values of v_0 and t_0 satisfy the above indifference condition and the limit version of first order conditions (3.7) and (3.8), given by

$$\pi'(t_0)(v_0 - t) + \pi(t_0) - \phi(v_0) = 0.$$

We have $t_0 > r^*$ and $\pi(t_0) > \phi(v_0)$. When α is arbitrarily close to 0, the incentives for informed buyers not to pretend to be uninformed are provided by raising the take-it-leave-it offer to an unlikely uninformed buyer above r^*, and not selling to uninformed buyers even when the profit from doing so exceeds virtual valuations of informed buyers.

[10]The limit of $\chi(v_-, v_+)$ as α goes to 0 and v_- and v_+ shrink to the same point of v_0 is $F^{n-1}(v_0)$. That is, when all other bidders are almost surely informed, a deviating informed bidder will be the only buyer in the equal priority pool and will get the good with probability one if all other bidders (who are informed) have valuation below v_0.

In the opposite limit of $\alpha = 1$, bidders are surely uninformed, and the revenue from the optimal equal priority auction converges to the revenue from a take-it-or-leave-it offer r^*. By (3.8), the seller no longer distorts t to provide incentives for informed buyers. From (3.7), the upper-bound of the pooling interval converges to $\overline{r}(r^*)$, satisfying

$$\phi(\overline{r}(r^*)) = \pi(r^*),$$

as the need for the seller to provide incentives for informed buyers with valuations just above the upper-bound becomes second order. From the binding constraint (3.4), the lower-bound of the pooling interval becomes r^*.[11] This is to prevent an unlikely informed buyer with a valuation equal to the lower bound from pretending to be uninformed, as the buyer has almost zero chance of winning the auction with the limit reserve price r_1 satisfying (3.9)

$$-\phi(r_1)f(r_1) = \pi(r^*)f(r^*).$$

As long as α is strictly less than 1, however, the auction is what provides incentives for informed buyers with valuations just below the lower bound of the interval not to pretend to be uninformed.

3.2. **Equilibrium mechanisms.** We use Lagrangian relaxation to show that an optimal equal-priority auction provides the seller the highest expected revenue among all direct mechanisms.

Recall that a direct mechanism δ consists of a series of functions $(q^\epsilon_m, p^\epsilon_m)^{n-1}_{m=0}$ and $(q^\mu_m, p^\mu_m)^n_{m=1}$. We first use the assumption that $\pi(\cdot)$ is strictly concave to simplify the optimal design problem. We show that replacing offers $p^\mu_m(v)$ to uninformed buyers with a single offer reduces the deviation payoff for informed from pretending to be uninformed, and improves the seller's revenue from uninformed buyers due to concavity of $\pi(\cdot)$.

Lemma 4. *If $\pi(\cdot)$ is strictly concave, then in any optimal direct mechanism, $p^\mu_m(v)$ is independent of m and v.*

Using Lemma 4, we denote the constant price offered to the uninformed as p^μ. Define

$$Q^\mu = \sum_{m=0}^{n-1} B(m; n-1, \alpha)\mathbb{E}_v\left\{q^\mu_{m+1}(v)\right\}$$

to be the total probability of an offer expected by an uninformed buyer (or a deviating informed bidder). As we have shown in Theorem 1 the revenue maximizing direct mechanism can be found by choosing a feasible mechanism δ that supports a trading probability for the uninformed Q^μ and a non-decreasing trading probability function $Q^\epsilon(\cdot)$ that maximizes

$$n(1-\alpha)\int_0^1 \{Q^\epsilon(w)\,\phi(w)f(w)dw\} + n\alpha Q^\mu \pi(p^\mu)$$

subject to

$$\int_0^w Q^\epsilon(x)\,dx \geq Q^\mu \max\left[w - p^\mu, 0\right] \tag{3.10}$$

for all w.

[11]The limit of $\chi(v_-, v_+)$ as α goes to 1 is $1/n$, as an unlikely informed buyer will surely face $n-1$ uninformed buyers in the equal priority pool after pretending to be uninformed.

Let $\lambda(\cdot)$ be an arbitrary non-negative Lagrangian function from $[0,1]$ into $\mathbb{R}$. The relaxed problem is to maximize

$$n(1-\alpha)\int_0^1 \{Q^\epsilon(w)\,\phi(w)f(w)dw\} + n\alpha Q^\mu \pi(p^\mu)$$
$$+\int_0^1 \lambda(w)\left\{\int_0^w Q^\epsilon(x)\,dx - Q^\mu \max[w-p^\mu,0]\right\}dw,$$

again by choosing $(q_m^\epsilon, p_m^\epsilon)_{m=0}^{n-1}$, $(q_m^\mu)_{m=1}^{n}$, and p^μ such that the feasibility constraint (2.6) is satisfied, and $Q^\epsilon(\cdot)$ is non-decreasing.

The above problem can have different solutions depending on the choice of $\lambda(\cdot)$. It is well known that the solution to the relaxed problem is an upper bound on the solution to the full problem no matter what the Lagrangian function.[12] The method of proof is to try to find a function $\lambda(\cdot)$ such that the solution to the relaxed problem is an equal priority auction. Since the equal priority auction yields an upper bound on the seller's payoff in the full problem, and since it satisfies all the constraints in the full problem, it must be a solution to the full problem.

To see how we came up with the multiplier function $\lambda(\cdot)$, use integration by parts and rewrite the Lagrangian as

$$\sum_{m=0}^{n-1} B(m;n-1,\alpha)\int_0^1 \left\{n(1-\alpha)\phi(w)f(w) + \int_w^1 \lambda(x)dx\right\} Q_m^\epsilon(w)dw$$
$$+\sum_{m=0}^{n-1} B(m;n-1,\alpha)\left(n\alpha\pi(p^\mu) - \int_0^1 \lambda(w)\max[w-p^\mu,0]dw\right) Q_{m+1}^\mu.$$

We want to choose $\lambda(\cdot)$ to have the following properties: (i) It takes value of 0 outside of $[v_-, v_+]$ so that the constraint (3.10) is slack. (ii) It takes non-negative values on $[v_-, v_+]$ such that the value of the expression in the first bracket in the above Lagrangian is constant, so that it is point wise maximizing to have constant $Q_m^\epsilon(w)$ for all $w \in [v_-, v_+]$. (iv) The constant value of the expression in the first bracket in the above Lagrangian matches the constant value of the expression in the second bracket, so that it is point wise maximizing to treat informed buyers with valuations in the pooling interval the same as uninformed buyers in terms of allocation. (iv) The value of the expression in the first bracket is greater than that in the second bracket for $w > v_+$ and smaller for $w < v_-$, so that informed buyers have higher priorities than uninformed buyers if their valuations are higher than v_+ and lower priorities if their valuations are lower than v_-.

Theorem 5. *Suppose that $\pi(\cdot)$ is strictly concave. Then, a revenue maximizing equal priority auction is a revenue maximizing direct mechanism.*

Putting together Theorems 5 and 1, we have shown that when $\pi(\cdot)$ is concave, the outcome of a symmetric equilibrium of the game $\mathcal{G}(\alpha)$ where uninformed buyers babble corresponds to an optimal equal priority auction. Conversely, once we solve for the revenue maximizing equal priority auction, we can construct a password mechanism to support a symmetric equilibrium of the game. Since equal priority auctions are relatively straightforward to describe and optimize over, we believe our

[12]The solution to the original problem is feasible, so the integral in the payoff to the relaxed problem is non-negative. In turn, the solution to the original problem gives a lower payoff in the relaxed problem than the solution to the relaxed problem itself.

result provides a simple characterization of equilibrium outcomes of the unobserved mechanism design game in the important class of uncommunicative messaging by uninformed buyers.

The relative simplicity of optimal equal priority auctions also allows us to understand welfare implications of unobserved mechanism design. The seller is of course worse off compared to when all buyers are informed, as unobservability reduces the power of commitment necessary for standard optimal auctions. This means that the seller has incentives to 'educate' buyers about his mechanism. But such attempt would be thwarted so long as the commitments in the mechanism remain unverifiable.

When all n buyers are informed, they face the standard optimal reserve price of r^*. In a symmetric uncommunicative equilibrium of the unobserved mechanism design game $\mathcal{G}(\alpha)$, the seller sets $r < r^*$, so an informed buyer with a valuation between r and r^* is better off than when there are no uninformed buyers around. Informed buyers with higher valuations are affected by the presence of uninformed and uncommunicative buyers in two opposing ways: they can win the auction even though some uninformed buyer has a higher valuation, but they may also lose to an uninformed with a lower valuation. The net effect is generally ambiguous, but we can show that informed buyers with sufficiently high valuations benefit from having uninformed buyers around if the number of buyers is sufficiently large.[13]

For uninformed buyers, the relevant welfare comparison question is how they are affected by the presence of informed buyers. If there are no informed buyers, uninformed buyers have an equal chance of receiving a take-it-or-leave-it offer equal to r^*. Since in a symmetric uncommunicative equilibrium of $\mathcal{G}(\alpha)$ the seller sets the take-it-or-leave-it offer t strictly above r^*, an uninformed buyer with a valuation w just above r^* is worse off in equilibrium than when there are no informed buyers around. For uninformed buyers with higher valuations, they have a higher priority than informed buyers with valuations below v_-, which makes them better off in equilibrium, but lose out to informed buyers with valuations above v_+. The net effect is again ambiguous, but we can show that uninformed buyers are all worse off in equilibrium than when there are no informed buyers if the number of buyers is sufficiently large.[14]

---

[13]To see this, note that

$$U^{\epsilon}(1) = \int_r^1 Q^{\epsilon}(w)dw > \int_{v_+}^1 ((1-\alpha)F(w)+\alpha)^{n-1}dw.$$

The above is greater than $\int_{r^*}^1 F^{n-1}(w)dw$ when n is sufficiently large, because by integration by parts, it is implied by

$$(1-\alpha)\int_{v_+}^1 ((1-\alpha)F(w)+\alpha)^{n-2}f(w)wdw < \int_{r^*}^1 F^{n-2}(w)f(w)wdw,$$

which is true for large enough n by using another integration by parts.

[14]To see this, note that

$$U^{\mu}(1) = \chi(v_-, v_+)(1-t) < ((1-\alpha)F(v_+)+\alpha)^{n-1}(1-r^*).$$

The above is less than $(1-r*)/n$ when n is sufficiently large. The payoff functions are piece wise linear, an uninformed buyer with any valuation is worse off in equilibrium.

4. Discussion

We have assumed that the objective of the seller is to make a single take it or leave it offer. If this offer is rejected, which it will sometimes be if it is made to an uninformed bidder, the game ends without trade. For the auction among the informed buyers this is without loss, since the winner of the auction always wants to accept the offer when they win the auction. For the uninformed this assumption is unrealistic. Once the seller learns who the uninformed buyers are, the seller is likely to approach them in sequence with offers. One question is how this might change if the seller could follow up a rejection by making an offer to one of the other bidders.

A general approach to unobserved mechanisms is to model the output of a mechanism as an "algorithm," which is a sequence of take-it-or-leave-it offers and the identities of the buyers to whom the offers are made. As in the present paper, the seller first makes a commitment in terms of how a particular algorithm is chosen in response to the messages sent by the buyers, who however may not observe it. It is straightforward to generalize the analysis in the present paper to the case in which algorithms are restricted to at most one take-it-or-leave-it offer for each buyer. The main insights are intact - an uninformed buyer receives an expected offer independent of the buyer's valuation, while informed buyers face a secret reserve price when they bid in an auction. We leave the characterization of unrestricted equilibrium algorithms to future research.

A more challenging question with multiple offers arises if uninformed buyers are unsure how many offers are made and how long they should wait for one. They might then choose to give up in a manner that depends on their waiting costs. We won't persue this here because it is obviously far more complex.

Yet we can make one observation. There will be at least one equilibrium in such a game that resolves exactly to our equal priority auction. In this equilibrium we just imagine that all buyer, informed and uninformed believe that only a single offer will be made at some arbitrary point in time. If they don't receive the offer at that point, they just don't expect to get one at all. Under those conditions, the seller can do no better than the equal priority auction because once an offer is rejected there won't be any buyers around to consider another one.

It is unclear whether alternative equilibrium with multiple offers might increase the seller's revenue. As always, multiple offer equilibrium improves the payoff to an informed buyer who wants to pretend to be uninformed. We defer this to future research since it is not clear at this point what is the best way to generalize to multiple offers.

Another sort of 'partial equilibrium' possibility is to assume that buyers wait around forever for offers. In this case the seller can do well by making a very high offer to each buyer in turn until all have rejected it. Then the seller could lower the offer slightly and continue to do this until some buyer accepts. This implements something that looks like a descending price auction.

The assumption that uninformed buyers will wait forever seems inconsistent at least with our view of why there are uninformed buyers in the first place - they have alternatives. Also, this is not the same as implementing a descending clock auction. Uninformed buyers won't understand how to bid in a descending clock auction for the same reason we described above. If they expected a descending clock auction and stopped optimally, the seller would add a password then change the rules to make them a counter offer.

4.1. **Concluding remarks.** In this paper we have considered a traditional mechanism design problem and modified it by assuming some buyers do not know the mechanism the seller is using. We show that, assuming uninformed buyers don't communicate any useful information, the seller's revenue optimal equilibrium can be implemented with an equal priority auction. This mechanism is new as far as we know. It lies nicely between the extremes of pure auction, which is best when the seller is sure everyone is informed, and a simple take it or leave price offer to a buyer chosen randomly.

One of the nice advantages of the equal priority auction is that it is parametric - all equal priority auctions can be described using only 4 parameters, which makes is easy to show existence. The parameters all lie in a compact set, and the payoff functions are integrals which depend continuously on the parameters.

The parametric representation makes it possible to do computations, and in principal, do empirical work. As we mentioned above, one of the implications of the the equal priority auctions is that the distribution of bids in the auction will be endogenous. In particular, it will be bi-modal with high and low bids while intermediate valuation bidders trade at a fixed price. This is something like what happens on eBay, though eBay auctions differ in many ways from what we have modeled here.

Perhaps a restrictive assumption we use is that buyers are either fully informed or fully uninformed. A more reasonable assumption might be that buyers have partial information about commitments. For example, we could assume that some buyers may only be able to understand commitments to actions based on their own messages, but not commitments that depend on the messages of others. If all buyers have this type of partial information, then there is an equilibrium in which the seller implements the optimal auction of Myerson (1981) through a first-price sealed bid auction. This corresponds to the main result of Akbarpour and Li (2018), who frame the issue of partial observability in terms of limited commitment by the seller. When buyers have differential information about the seller's commitments - for example, if buyers either fully observe the seller's commitment or only observe the part based on their own message - we nonetheless believe that our basic insight could be extended to this kind of assumption. Yet we are reluctant to pursue without a better model of what buyers can and cannot understand.

5. Appendix: Omitted Proofs

Proof of Theorem 1.

Proof. Recall that a perfect Bayesian equilibrium in $\mathcal{G}(\alpha)$ is given by some mixture ψ for the seller, and strategy rules $\left(\sigma_i(\cdot,\epsilon,\gamma),\sigma_i(\cdot,\mu)\right)_{i=1}^N$ for informed and uninformed buyers respectively. These formulas can be reduced using the two assumptions in the theorem - symmetry and babbling by uninformed buyers. Symmetry means that all informed buyers use the same strategy, so we drop the subscript i from $\sigma_i(\cdot,\epsilon,\gamma)$. Since uninformed buyers babble, their messages won't affect any outcomes can be implemented without these message. Therefore we can ignore the message strategy $\sigma_i(\cdot,\mu)$ of the uninformed entirely.

Symmetry also means all informed buyers that send the same message are treated the same way in each mechanism $\gamma = \{\mathcal{M}, (p_i, q_i)_{i=1}^N\}$ in the support of ψ. We reorder the n buyers such that the first $n-m$ of them are informed, and use the same permutation device ρ in direct mechanisms on bids from informed buyers to rewrite γ. Denote as $\tilde{q}_m^{\mu}(b,\gamma)$ the probability with which an offer $\tilde{p}_m^{\mu}(b,\gamma)$ is made to an uninformed buyer given that there are m uninformed buyers and the profile of $n-m$ bids from informed buyers is $b = \{b_1, \ldots, b_{n-m}\}$. The function $\tilde{q}_m^{\epsilon}(b,\gamma)$ gives the probability with which an offer $\tilde{p}_m^{\epsilon}(b,\gamma)$ is made to the buyer with the first value in b, given that there are m uninformed buyers and the other $n-m-1$ informed buyers bid $b_{-1} = \{b_2, \ldots, b_{n-m}\}$. Feasibility of γ requires

$$\sum_{i=1}^{n-m} \tilde{q}_m^{\epsilon}\left(\rho_m^i(b), \gamma\right) + m\tilde{q}_m^{\mu}(b,\gamma) \leq 1$$

for all m.

The equilibrium condition (2.2) for an informed buyer with valuation v_1 in mechanism γ can now be rewritten as

$$\begin{aligned}&\sum_{m=0}^{n-1} B(m;n-1,\alpha)\mathbb{E}_{v_2,\ldots,v_{n-m}}\left\{\tilde{q}_m^{\epsilon}\left(\sigma(v_1,\epsilon,\gamma),\ldots,\sigma(v_{n-m},\epsilon,\gamma),\gamma\right)\right.\\&\qquad\qquad\left.\cdot\max\left[v_1-\tilde{p}_m^{\epsilon}\left(\sigma(v_1,\epsilon,\gamma),\ldots,\sigma(v_{n-m},\epsilon,\gamma),\gamma\right),0\right]\right\}\\&\geq\sum_{m=0}^{n-1} B(m;n-1,\alpha)\mathbb{E}_{v_2,\ldots,v_{n-m}}\left\{\tilde{q}_m^{\epsilon}\left(b',\ldots,\sigma(v_{n-m},\epsilon,\gamma),\gamma\right)\right.\\&\qquad\qquad\left.\cdot\max\left[v_1-\tilde{p}_m^{\epsilon}\left(b',\ldots,\sigma(v_{n-m},\epsilon,\gamma),\gamma\right),0\right]\right\},\end{aligned}$$

for all $b' \in \mathcal{M}$. The expected payoff for the informed buyer from pretending to be uninformed, and also the equilibrium payoff for an uninformed buyer with the same valuation, is given by

$$\begin{aligned}&\sum_{m=0}^{n-1} B(m;n-1,\alpha)\mathbb{E}_{v_2,\ldots,v_{n-m}}\left\{\tilde{q}_{m+1}^{\mu}\left(\sigma(v_2,\epsilon,\gamma),\ldots,\sigma(v_{n-m},\epsilon,\gamma),\gamma\right)\right.\\&\qquad\qquad\left.\cdot\max\left[v_1-\tilde{p}_{m+1}^{\mu}\left(\sigma(v_2,\epsilon,\gamma),\ldots,\sigma(v_{n-m},\epsilon,\gamma),\gamma\right),0\right]\right\}.\end{aligned}$$

We now define a direct mechanism $\delta^* = \left\{[0,1], (q_m^{\epsilon}, p_m^{\epsilon})_{m=0}^{n-1}, (q_m^{\mu}, p_m^{\mu})_{m=1}^{n}\right\}$, where for each $m = 1, \ldots, n$ and each $v = (v_1, \ldots, v_n)$,

$$\begin{aligned}q_m^{\mu}(v) &= \tilde{q}_m^{\mu}(\sigma(v_1,\epsilon,\gamma),\ldots,\sigma(v_{n-m},\epsilon,\gamma),\gamma),\\p_m^{\mu}(v) &= \tilde{q}_m^{\mu}(\sigma(v_1,\epsilon,\gamma),\ldots,\sigma(v_{n-m},\epsilon,\gamma),\gamma);\end{aligned}$$

and for each $m = 0, \ldots, n-1$ and each $v = (v_1, \ldots, v_n)$,

$$\begin{aligned}q_m^{\epsilon}(v) &= \tilde{q}_m^{\epsilon}(\sigma(v_1,\epsilon,\gamma),\ldots,\sigma(v_{n-m},\epsilon,\gamma),\gamma),\\p_m^{\epsilon}(v) &= \tilde{q}_m^{\epsilon}(\sigma(v_1,\epsilon,\gamma),\ldots,\sigma(v_{n-m},\epsilon,\gamma),\gamma)\end{aligned}$$

if $\tilde{p}_m^{\epsilon}(\sigma(v_1,\epsilon,\gamma),\ldots,\sigma(v_{n-m},\epsilon,\gamma),\gamma) \geq v_1$; and

$$q_m^{\epsilon}(v) = 0,\ p_m^{\epsilon}(v) = v_1$$

if $\tilde{p}_m^{\epsilon}(\sigma(v_1,\epsilon,\gamma),\ldots,\sigma(v_{n-m},\epsilon,\gamma),\gamma) > v_1$.

It is now easy to see that the direct mechanism defined above is incentive compatible in that truthful bidding is a Bayesian Nash equilibrium among informed buyers. The feasibility constraint (2.6) is satisfied. The direct mechanism achieves the same revenue as γ.

In any equilibrium of $\mathcal{G}(\alpha)$, the seller's revenue is the same for each realization γ in the support of ψ. Thus the expected revenue $R(\delta^*)$ from δ^* achieves the same equilibrium revenue for the seller. Further, there is no incentive compatible direct mechanism δ that achieves a strictly revenue $R(\delta)$ than $R(\delta^*)$. If there were, then given that uninformed buyers babble and informed buyers can condition their strategies on the mechanism, the seller would deviate to the more profitable direct mechanism δ, which contradicts the equilibrium condition (2.5) for the seller.

The reverse direction of Theorem 1 follows by constructing a symmetric equilibrium of $\mathcal{G}(\alpha)$ using password mechanisms derived from an optimal direct mechanism $\delta^* = \{[0,1], (q^{\epsilon}_m, p^{\epsilon}_m)_{m=0}^{n-1}, (q^{\mu}_m, p^{\mu}_m)_{m=1}^{n}\}$. The seller's equilibrium strategy ψ is a mixture ψ over password mechanisms $\gamma(\zeta)$, where each ζ is a realization ζ of a uniform random variable on $[0,1]$. Each password mechanism $\gamma(\zeta)$ has message space $[0,1]^2$, with a typical message $b = (z, v)$. Given a realized password mechanism $\gamma(\zeta)$, the equilibrium strategy of an informed buyer i with valuation v_i is to send message (ζ, v_i). The equilibrium strategy of an uninformed buyer with any valuation is a pair of independent and random draws from the uniform distribution over $[0,1]$. For each password mechanism $\gamma(\zeta)$, the trading probabilities and offers $(\hat{q}_i, \hat{p}_i)_{i=1}^N$ are derived from δ^* as follows. For each profile of messages $(b_1, \ldots, b_n) = ((z_1, v_1), \ldots, (z_n, v_n))$, let $m = \#\{i : z_i \neq \zeta\}$, and reorder the buyers $z_i = \zeta$ for each $i = 1, \ldots, n-m$. Define

$$\hat{q}_i(b, \zeta) = q^{\epsilon}_m(v),\ \hat{p}_i(b, \zeta) = p^{\epsilon}_m(v)$$

for each $i = 1, \ldots, n-m$, and

$$\hat{q}_i(b, \zeta) = q^{\mu}_m(v),\ \hat{p}_i(b, \zeta) = p^{\mu}_m(v)$$

for each $i = n-m+1, \ldots, n$.

It is straightforward to verify that the strategies of the seller, informed and uninformed buyers form a perfect Bayesian equilibrium. In particular, given that the seller ignores any valuation report by buyer i when i does not match the realized password ζ, it is an optimal response for uninformed buyers report a random number from $[0,1]$ as his valuation since he does not observe the password. Conversely, given that uninformed buyers report a random number as their valuation, it is optimal for the seller to commit to mechanisms that ignore any valuation report by a buyer who does not match the realized password. □

Proof of Lemma 2.

Proof. First, an informed buyer with $v < r$ never wins the auction, and thus the expected payoff is 0, matching $U^{\epsilon}(v)$ in (3.1) and (3.3) for $v < r$.

Second, an informed buyer with $v \in [r, v_-)$ wins the auction only when $m = 0$ and all $n-1$ other informed buyers have valuation at most v, pays the maximum of r and the second highest valuation. Thus, the expected payoff is

$$v(1-\alpha)^{n-1}F^{n-1}(v) - \left(r(1-\alpha)^{n-1}F^{n-1}(r) + \int_r^v w \ d\left((1-\alpha)^{n-1}F^{n-1}(w)\right) \right).$$

By integration by parts, the above matches $U^{\epsilon}(v)$ in (3.1) and (3.3) for $v \in [r, v_-)$.

Third, an informed buyer with $v \in [v_-, v_+]$ wins the auction when $m = 0$ and all $n-1$ other informed buyers have valuation at most v_-, and pays the maximum of r and the second highest valuation. The contribution of this event to the buyer's expected payoff is

$$\begin{aligned} & v(1-\alpha)^{n-1}F^{n-1}(v_-) - \left(r(1-\alpha)^{n-1}F^{n-1}(r) + \int_r^{v_-} w \; d\left((1-\alpha)^{n-1}F^{n-1}(w)\right) \right) \\ =& U^\epsilon(v_-) + (v - v_-)(1-\alpha)^{n-1}F^{n-1}(v_-). \end{aligned}$$

The buyer also wins the auction with probability $1/(m+k+1)$ when there are m uninformed buyers, all $n-m-1$ other informed buyers have valuation at most v_+, and $m+k$ is at least 1 (where k is the number of informed buyers with valuation on $[v_-, v_+]$), and pays v_-. The contribution of this event to the buyer's expected payoff is

$$(v - v_-)\left(\chi(v_-, v_+) - (1-\alpha)^{n-1}F^{n-1}(v_-)\right).$$

The sum of the above two expressions matches $U^\epsilon(v)$ in (3.1) and (3.3) for $v \in [v_-, v_+]$.

Fourth, for an informed buyer with $v > v_+$ who wins the auction, he pays the maximum of the second highest bid and the reserve price. When the second highest bid is below v_-, which implies that $m = k = 0$, the reserve price is r, and the contribution to the expected payoff is

$$U^\epsilon(v_-) + (v - v_-)(1-\alpha)^{n-1}F^{n-1}(v_-).$$

When the second highest bid is between v_- and v_+, which implies that $m + k \geq 1$, the reserve price is $(v_- + v_+(m+k))/(m+k+1)$, and the contribution to the expected payoff is

$$\begin{aligned} & \sum_{m=0}^{n-1} B(m; n-1, \alpha) \sum_{k=0}^{n-1-m} B_k^{n-1-m}(v_-, v_+) \left(v - \frac{v_- + v_+(m+k)}{m+k+1} \right) \\ & \qquad - (1-\alpha)^{n-1}F^{n-1}(v_-)(v - v_-) \\ =& (v - v_+)((1-\alpha)F(v_+) + \alpha)^{n-1} + (v_+ - v_-)\chi(v_-, v_+) - (1-\alpha)^{n-1}F^{n-1}(v_-)(v - v_-). \end{aligned}$$

When the second highest bid w is above v_+, which occurs with probability

$$\sum_{m=0}^{n-1} B(m; n-1, \alpha)(F^{n-1-m}(w) - F^{n-1-m}(v_+)),$$

the buyer pays this bid, and so by integration by parts the contribution to the expected payoff is

$$\begin{aligned} & \int_{v_+}^{v} \sum_{m=0}^{n-1} B(m; n-1, \alpha)(F^{n-1-m}(w) - F^{n-1-m}(v_+))dw \\ =& \int_{v_+}^{v} \sum_{m=0}^{n-1} B(m; n-1, \alpha)F^{n-1-m}(w)dw - (v - v_+)((1-\alpha)F(v_+) + \alpha)^{n-1}. \end{aligned}$$

The sum of the three expressions for the contributions to the expected payoff matches $U^\epsilon(v)$ in (3.1) and (3.3) for $v > v_+$. □

Proof of Lemma 3.

Proof. Define

$$D(r,t,v_-,v_+) = U^\epsilon(v_-) - U^\mu(v_-) = \int_r^{v_-} (1-\alpha)^{n-1}F^{n-1}(w)dw - \chi(v_-,v_+)(v_- - t),$$

and let R be the revenue from the equal-priority auction. We have

$$\frac{\partial D}{\partial r} = -(1-\alpha)^{n-1}F^{n-1}(r);$$
$$\frac{\partial R}{\partial r} = -n(1-\alpha)^n F^{n-1}(r)\phi(r)f(r)$$

and

$$\frac{\partial D}{\partial t} = \chi(v_-,v_+);$$
$$\frac{\partial R}{\partial t} = n\alpha\chi(v_-,v_+)\pi'(t).$$

If $v_- < v_+$, or if $v_- = v_+$ and $dv_- < 0$, we have

$$\frac{\partial\chi(v_-,v_+)}{\partial v_-} = \frac{(1-\alpha)f(v_-)}{(1-\alpha)(F(v_+)-F(v_-))+\alpha}\left(\chi(v_-,v_+) - ((1-\alpha)F(v_-))^{n-1}\right);$$
$$\frac{\partial D}{\partial v_-} = (1-\alpha)^{n-1}F^{n-1}(v_-) - \chi(v_-,v_+) - \frac{\partial\chi(v_-,v_+)}{\partial v_-}(v_- - t);$$
$$\frac{\partial R}{\partial v_-} = n(1-\alpha)((1-\alpha)^{n-1}F^{n-1}(v_-) - \chi(v_-,v_+))\phi(v_-)f(v_-)$$
$$+ n((1-\alpha)(\pi(v_-)-\pi(v_+)) + \alpha\pi(t))\frac{\partial\chi(v_-,v_+)}{\partial v_-}.$$

If $v_- < v_+$, or if $v_- = v_+$ and $dv_+ > 0$, we have

$$\frac{\partial\chi(v_-,v_+)}{\partial v_+} = \frac{(1-\alpha)f(v_+)}{(1-\alpha)(F(v_+)-F(v_-))+\alpha}\left(((1-\alpha)F(v_+)+\alpha)^{n-1} - \chi(v_-,v_+)\right);$$
$$\frac{\partial D}{\partial v_+} = -\frac{\partial\chi(v_-,v_+)}{\partial v_+}(v_- - t);$$
$$\frac{\partial R}{\partial v_+} = n(1-\alpha)\left(\chi(v_-,v_+) - ((1-\alpha)F(v_+)+\alpha)^{n-1}\right)\phi(v_+)f(v_+)$$
$$+ n((1-\alpha)(\pi(v_-)-\pi(v_+)) + \alpha\pi(t))\frac{\partial\chi(v_-,v_+)}{\partial v_+}.$$

If $v_- = v_+ = \hat{v}$, we have

$$\frac{d\chi(\hat{v},\hat{v})}{d\hat{v}} = \frac{(1-\alpha)f(\hat{v})}{\alpha}\left(((1-\alpha)F(\hat{v})+\alpha)^{n-1} - ((1-\alpha)F(\hat{v}))^{n-1}\right);$$
$$\frac{\partial D}{\partial \hat{v}} = (1-\alpha)^{n-1}F^{n-1}(\hat{v}) - \chi(\hat{v},\hat{v}) - \frac{d\chi(\hat{v},\hat{v})}{d\hat{v}}(\hat{v}-t);$$
$$\frac{\partial R}{\partial \hat{v}} = n(1-\alpha)\left((1-\alpha)^{n-1}F^{n-1}(\hat{v}) - ((1-\alpha)F(\hat{v})+\alpha)^{n-1}\right)\phi(\hat{v})f(\hat{v}) + n\alpha\pi(t)\frac{d\chi(\hat{v},\hat{v})}{d\hat{v}}$$
$$= n(1-\alpha)f(\hat{v})\left(((1-\alpha)F(\hat{v})+\alpha)^{n-1} - (1-\alpha)^{n-1}F^{n-1}(\hat{v})\right)(\pi(t)-\phi(\hat{v})).$$

Let (r,t,v_-,v_+) be an optimal equal-priority auction. We first show that it is interior.

Suppose that $r = t < v_- \le v_+$. Recall that $U^\epsilon(v)$ is strictly convex for $v \in (r, v_-)$ while $U^\mu(v)$ is linear for $v \in (t, v_-)$. Since $Q^\epsilon(v)$ has an upward jump at $v = v_-$, we have $U^\epsilon(v_-) < U^\mu(v_-)$, violating the critical bidding condition (3.4).

Suppose that $r < t = v_- \le v_+$. We have $U^\epsilon(v_-) > U^\mu(v_-) = 0$, and so the critical bidding condition (3.4) is slack. Since $r < t$, we have $r < r^*$ or $t > r^*$, or both. If $r < r^*$, then by raising r marginally, the seller could increase the revenue because $\phi(r) < 0$ implies $\partial R/\partial r > 0$. If $t > r^*$, then by lowering t marginally, the seller could increase the revenue because $\pi'(t) < 0$ implies $\partial R/\partial t < 0$. With the critical bidding condition (3.4) slack, we have a contradiction to the assumption of optimality.

Suppose that $r = t = v_- \le v_+$. If $r = t < r^*$, then by raising t marginally, the seller relaxes the critical bidding condition (3.4) because $\partial D/\partial t > 0$, and increases the revenue because $\pi'(t) > 0$ implies $\partial R/\partial t > 0$. If $r = t > r^*$, then by lowering r marginally, the seller relaxes the critical bidding condition (3.4) because $\partial D/\partial r < 0$, and increases the revenue because $\phi(r) > 0$ implies $\partial R/\partial r < 0$. If $r = t = r^* = v_-$, then by lowering r marginally, the seller relaxes the critical bidding condition (3.4) because $\partial D/\partial r < 0$, without changing the revenue because $\partial R/\partial r = 0$. With (3.4) slack, the seller could then increase the revenue by either further raising v_- marginally if $v_- = r^* < v_+$, because $\phi(v_-) = 0$ implies $\partial R/\partial v_- > 0$, or by raising both v_- and v_+ by the same infinitesimal amount if $v_- = v_+ = r^*$, because $\partial R/\partial \hat{v} > 0$ when $\hat{v} = r^*$. In each case, we have a contradiction to the assumption of optimality.

Suppose that $r < t < v_- = v_+ = \hat{v}$. We have $\partial D/\partial \hat{v} < 0$ and $\partial R/\partial \hat{v} < 0$ has the same sign as $\pi(t) - \phi(\hat{v})$. Thus, $\pi(t) > \phi(\hat{v})$: otherwise, by decreasing v_- and v_+ by the same marginal amount, the seller relaxes the critical bidding condition (3.4) without decreasing the revenue, which would then allow the seller to increase the revenue by either raising r or lowering t, as $r < t$ implies $r < r^*$ or $t > r^*$, or both. Since $\phi(1) = 1$, this implies that $\hat{v} < 1$. Now, consider perturbing the equal priority auction by reducing v_- from $\hat{v}$ and raising v_+ from $\hat{v}$ such that

$$-(\chi(\hat{v},\hat{v}) - (1-\alpha)^{n-1}F^{n-1}(\hat{v}))dv_- = (((1-\alpha)F(\hat{v}) + \alpha)^{n-1} - \chi(\hat{v},\hat{v}))dv_+.$$

By construction,

$$\frac{\partial \chi(\hat{v},\hat{v})}{\partial v_-} = \frac{\partial \chi(\hat{v},\hat{v})}{\partial v_+}.$$

This implies that the critical bidding condition (3.4) is relaxed, because

$$\frac{\partial D}{\partial v_-}dv_- + \frac{\partial D(\hat{v})}{\partial v_+}dv_+ = ((1-\alpha)^{n-1}F^{n-1}(\hat{v}) - \chi(\hat{v},\hat{v}))dv,$$

which is strictly negative. The seller's revenue is unchanged, because

$$\begin{aligned}&\frac{\partial R}{\partial v_-}dv_- + \frac{\partial R}{\partial v_+}dv_+\\ =&n(1-\alpha)f(\hat{v})\left(\chi(\hat{v},\hat{v}) - (1-\alpha)^{n-1}F^{n-1}(\hat{v})\right)(\pi(t) - \phi(\hat{v}))dv_-\\ &+ n(1-\alpha)f(\hat{v})\left(((1-\alpha)F(\hat{v}) + \alpha)^{n-1} - \chi(\hat{v},\hat{v})\right)(\pi(t) - \phi(\hat{v}))dv_+,\end{aligned}$$

which is equal to 0 by construction. The seller could now increase the revenue by either raising r or lowering t, as $r < t$ implies $r < r^*$ or $t > r^*$, or both. This contradicts the assumption of optimality.

Now, we establish the first-order conditions stated in the lemma. To begin, the critical bidding condition (3.4) binds at any optimal equal-priority auction.

Otherwise, since $r < t$ implies that $r < r^*$ or $t > r^*$, or both, the seller could increase the revenue by either raising r or lowering t, a contradiction to the assumed optimality. Further, $r < r^* < t$. Otherwise, if $r^* \leq r < t$, the seller could relax (3.4) by lowering r marginally without decreasing the revenue, which then would allow the seller to increase the revenue by lowering t. Similarly, if $r < t \leq r^*$, the seller could relax (3.4) by raising t marginally without decreasing the revenue, which then would allow the seller to increase the revenue by raising r. Finally, $\pi(t) > \phi(v_+)$. Otherwise, by lowering v_+ marginally, the seller relaxes (3.4) because $\partial D/\partial v_+ < 0$, and increases the revenue, as $\partial R/\partial v_+$ has the same sign as

$$\begin{aligned} &\alpha(\pi(t) - \phi(v_+)) + (1-\alpha)(\pi(v_-) - \pi(v_+)) - \phi(v_+)(F(v_+) - F(v_-)) \\ =&\alpha(\pi(t) - \phi(v_+)) - \int_{v_-}^{v_+} (\phi(v_+) - \phi(w))f(w)dw \\ <&\alpha(\pi(t) - \phi(v_+)), \end{aligned}$$

contradicting the assumed optimality. Note that $\pi(t) > \phi(v_+)$ implies $v_+ < 1$.

To obtain (3.7), consider perturbations dv_- and dv_+, while keeping r and t unchanged. An optimality condition is that

$$\frac{\partial R}{\partial v_-}dv_- + \frac{\partial R}{\partial v_+}dv_+ = 0,$$

for all perturbations dv_- and dv_+ satisfying

$$\frac{\partial D}{\partial v_-}dv_- + \frac{\partial D}{\partial v_+}dv_+ = 0.$$

Thus we have

$$\frac{\partial R/\partial v_-}{\partial D/\partial v_-} = \frac{\partial R/\partial v_+}{\partial D/\partial v_+}.$$

Using the expressions for $\chi(v_-, v_+)$, $\partial\chi(v_-, v_+)/\partial v_-$ and $\partial\chi(v_-, v_+)/\partial v_+$, straightforward algebra lead us to the first-order condition (3.7) for an optimal equal-priority auction with respect to v_- and v_+. Note that (3.7) implies that

$$\frac{\partial R/\partial v_+}{\partial D/\partial v_+} = -n(1-\alpha)(\phi(v_+) - \phi(v_-))f(v_-).$$

Next, to obtain (3.8), consider perturbations dt and dv_+. The resulting optimality condition is

$$\frac{\partial R/\partial t}{\partial D/\partial t} = \frac{\partial R/\partial v_+}{\partial D/\partial v_+}.$$

This gives the first order condition (3.8) with respect to t and v_+.

Lastly, to obtain (3.9), consider perturbations dr and dv_+, while keeping t and v_- unchanged. The resulting optimality condition is

$$\frac{\partial R/\partial r}{\partial D/\partial r} \geq \frac{\partial R/\partial v_+}{\partial D/\partial v_+},$$

and $r \geq 0$, with complementary slackness. This gives the first-order condition

$$-\phi(r)f(r) \leq (\phi(v_+) - \phi(v_-))f(v_-),$$

and $r \geq 0$, with complementary slackness. Note that $-\phi(0)f(0) = 1$. Since $\phi(v_+) < \pi(t)$ and $t > r^*$, we have $\phi(v_+) < \pi(r^*) < r^*$, while $v_- > t > r^*$. Thus,

$$(\phi(v_+) - \phi(v_-))f(v_-) = (\phi(v_+) - v_-)f(v_-) + 1 - F(v_-) < 1.$$

It follows that the optimal r is interior and so (3.9) holds. □

Proof of Lemma 4.

Proof. Fix a direct mechanism $\{q^\epsilon_m, p^\epsilon_m\}_{m=0}^{n-1}$ and $\{q^\mu_m, p^\mu_m\}_{m=1}^{n}$. Define $p^\mu \in [0,1]$ to be the expected offer to uninformed buyers, given by

$$\sum_{m=0}^{n-1} B(m; n-1, \alpha)\mathbb{E}_v\{q^\mu_{m+1}(v)(p^\mu - p^\mu_{m+1}(v))\} = 0.$$

Since $p^\mu_m(v) \in [0,1]$ for all v,

$$\sum_{m=0}^{n-1} B(m; n-1, \alpha)\mathbb{E}_v\{q^\mu_{m+1}(v)\} \max[w - p^\mu, 0]$$

$$= \sum_{m=0}^{n-1} B(m; n-1, \alpha)\mathbb{E}_v\left\{q^\mu_{m+1}(v) \max[w - p^\mu_{m+1}(v), 0]\right\}$$

for all $w \le \min p^\mu_m(v)$ and for all $w \ge \max p^\mu_m(v)$. Since $U^\mu(w)$ is convex in w, we have

$$U^\mu(w) \ge \sum_{m=0}^{n-1} B(m; n-1, \alpha)\mathbb{E}_v\{q^\mu_{m+1}(v)\} \max[w - p^\mu, 0]$$

for all w. Thus, replacing each all functions $\{p^\mu_m(\cdot)\}_{m=1}^n$ with p^μ reduces the deviation payoff of an informed buyer from pretending to be uninformed. The seller's revenue from uninformed buyers is

$$\sum_{m=1}^{n} B(m; n, \alpha)\mathbb{E}_v\left\{mq^\mu_m(v)\pi\left(p^\mu_m(v)\right)\right\} = n\alpha \sum_{m=0}^{n-1} B(m; n-1, \alpha)\mathbb{E}_v\left\{q^\mu_{m+1}(v)\pi\left(p^\mu_{m+1}(v)\right)\right\}.$$

The lemma then follows from the strict concavity of $\pi(\cdot)$. □

Proof of Theorem 5.

Proof. Suppose that $\{r, t, v_-, v_+\}$ is a revenue maximizing equal priority auction. By Lemma 3, the first order conditions (3.7)-(3.9) are satisfied. We construct a non-negatively valued multiplier function $\lambda(w)$ for all $w \in [0,1]$ such that the allocative rule $(q^\epsilon_m)_{m=0}^{n-1}$ and $(q^\mu_m)_{m=1}^{n}$, together with the offer to uninformed p^μ defined by $\{r, t, v_-, v_+\}$ solves the Lagrangian relaxation. By Lemma 2, the transfer rule we have specified for an equal priority auction supports a truthful bidding equilibrium among informed buyers. Thus we have found a direct mechanism $\left\{(q^\epsilon_m, p^\epsilon_m)_{m=0}^{n-1}, (q^\mu_m, p^\mu_m)_{m=1}^{n}\right\}$ that point-wise maximizes the Lagrangian.

For each $w \in [0,1]$, denote

$$K^\epsilon(w) = n(1-\alpha)\phi(w) + \int_w^1 \lambda(x)dx/f(w);$$

$$K^\mu = n\alpha\pi(p^\mu) - \int_0^1 \lambda(x)\max[x - p^\mu, 0]dx.$$

We can then rewrite the Lagrangian as

$$(1-\alpha)^{n-1}\int_0^1 K^\epsilon(w)Q_0^\epsilon(w)f(w)dw+\alpha^{n-1}K^\mu q_n^\mu$$
$$+\sum_{m=1}^{n-1}\left(\int_0^1 B(m;n-1,\alpha)K^\epsilon(w)Q_m^\epsilon(w)f(w)dw+B(m-1;n-1,\alpha)K^\mu Q_m^\mu\right),$$

where $Q_0^\epsilon(w)$ is the probability that an informed buyer with valuation w gets the good when all buyers are informed, and q_n^μ is the probability that each uninformed buyer gets the good when all buyers are uninformed.

Now we construct $\lambda(\cdot)$ as follows. Let $\lambda(w)=0$ for all $w\notin[v_-,v_+]$, and let

$$\begin{aligned}\lambda(w)=&n(1-\alpha)\frac{d}{dw}(f(w)(\phi(w)-\phi(v_+)))\\=&n(1-\alpha)(2f(w)+f'(w)(w-\phi(v_+)))\end{aligned}$$

for all $w\in(v_-,v_+)$, with $\lambda(v_-)$ and $\lambda(v_+)$ given by the corresponding limit from above and from below. Since by assumption $\pi(\cdot)$ is strictly concave, $f(w)\phi(w)$ is strictly increasing in w, and thus $\lambda(w)>0$ at any $w\in[v_-,v_+]$ such that $f'(w)\le 0$. By (3.7) we have $\phi(v_+)<\pi(t)<\pi(r^*)<r^*$. Since $w\ge v_->t>r^*$, we have $\lambda(w)>0$ at any $w\in[v_-,v_+]$ such that $f'(w)>0$. Thus, $\lambda(w)$ as constructed is non-negative for any w.

We will first show that $p^\mu=t$ maximizes the Lagrangian. For any $w\in[v_-,v_+]$, by construction

$$\int_w^1\lambda(x)dx=n(1-\alpha)f(w)(\phi(v_+)-\phi(w)).$$

Using integration by parts, we have

$$\begin{aligned}&\int_0^1\lambda(w)\max[w-p^\mu,0]dw\\=&-\int_{v_-}^{v_+}(w-p^\mu)\ d\left(\int_w^1\lambda(x)dx\right)\\=&n(1-\alpha)\left((v_--p^\mu)f(v_-)(\phi(v_+)-\phi(v_-))+\int_{v_-}^{v_+}f(w)(\phi(v_+)-\phi(w))dw\right)\\=&n(1-\alpha)\left((v_--p^\mu)f(v_-)(\phi(v_+)-\phi(v_-))+\phi(v_+)(F(v_+)-F(v_-))-(\pi(v_-)-\pi(v_+))\right).\end{aligned}$$

By (3.7), we have

$$K^\mu=n\alpha\phi(v_+)+n\alpha(\pi(p^\mu)-\pi(t))+(p^\mu-t)n(1-\alpha)f(v_-)(\phi(v_+)-\phi(v_-)).$$

The above is strictly concave in p^μ. By (3.8), it is maximized at $p^\mu=t$, and thus

$$K^\mu=n\alpha\phi(v_+).$$

The remainder of the proof establishes that the direct mechanism $(q_m^\epsilon)_{m=0}^{n-1}$, $(q_m^\mu)_{m=1}^n$, and $p^\mu=t$ defined by $\{r,t,v_-,v_+\}$ is a point-wise maximizer of the Lagrangian relaxation. For $w\in[v_-,v_+]$, we have

$$\frac{B(m;n-1,\alpha)}{n-m}K^\epsilon(w)=\frac{B(m-1;n-1,\alpha)}{m}K^\mu.$$

For all $w>v_+$, since $\pi(\cdot)$ is strictly concave,

$$K^\epsilon(w)=n(1-\alpha)\phi(w)>n(1-\alpha)\phi(v_+)=K^\epsilon(v_+),$$

and so

$$\frac{B(m;n-1,\alpha)}{n-m}K^{\epsilon}(w) > \frac{B(m-1;n-1,\alpha)}{m}K^{\mu}.$$

For all $w < v_-$,

$$\begin{aligned}K^{\epsilon}(w) &= n(1-\alpha)\phi(w) + \int_{v_-}^{v_+} \lambda(x)dx/f(w) \\ &= n(1-\alpha)(\phi(w) + f(v_-)(\phi(v_+) - \phi(v_-))/f(w)).\end{aligned}$$

We claim that

$$\phi(w) + \frac{f(v_-)(\phi(v_+) - \phi(v_-))}{f(w)} < \phi(v_+)$$

for all $w < v_-$, and thus $K^{\epsilon}(w) < K^{\epsilon}(v_+)$ and

$$\frac{B(m;n-1,\alpha)}{n-m}K^{\epsilon}(w) \leq \frac{B(m-1;n-1,\alpha)}{m}K^{\mu}.$$

To establish the claim, recall that in showing that the constructed multiplier function $\lambda(w)$ is positive for $w \in [v_-, v_+]$, we have proved that $f(w)(\phi(w) - \phi(v_+))$ is strictly increasing in w for all $w \geq \phi(v_+)$. This immediately implies that the claim holds for any $w \in [\phi(v_+), v_-)$. For $w < \phi(v_+)$, we have

$$f(w)(\phi(w)-\phi(v_+)) = f(w)(w-\phi(v_+))-(1-F(w)) < -(1-F(w)) < -(1-F(r^*)),$$

where the last inequality follows because $\phi(v_+) < \pi(t) < \pi(r^*) < r^*$, while

$$f(v_-)(\phi(v_+) - \phi(v_-)) < f(r^*)\phi(v_+) < f(r^*)r^*,$$

where the first equality comes from $f(w)(\phi(w) - \phi(v_+))$ being strictly increasing in w for all $w \geq \phi(v_+)$. The claim then follows from the definition of r^*.

To show that the direct mechanism defined by the equal-priority auction $\{r, v_-, v_+, t\}$ is a point-wise maximizer of the Lagrangian, we disaggregate $Q_m(w)$ and write the Lagrangian as

$$(1-\alpha)^{n-1}\int_0^1 K^{\epsilon}(w)Q_0^{\epsilon}(w)f(w)dw + \alpha^{n-1}K^{\mu}q_n^{\mu} +$$

$$\sum_{m=1}^{n-1}\mathbb{E}_v\left\{\frac{B(m;n-1,\alpha)}{n-m}\sum_{i=1}^{n-m}K^{\epsilon}(v_i)q_m^{\epsilon}(\rho_m^i(v)) + B(m-1;n-1,\alpha)K^{\mu}q_m^{\mu}(v)\right\}.$$

Fix any realized number m of uninformed buyers such that $1 \leq m \leq n-1$, and consider the last term in the above objective function. Suppose that for some realized valuation profile v we have $v_i > v_+$ for some $i = 1, \ldots, n-m$, but $q_m^{\mu}(v) > 0$. By (2.6), we can decrease $q_m^{\mu}(v)$ marginally by $dq_m^{\mu}(v) > 0$ and increase $q_m^{\epsilon}(\rho_m^i(v))$ by $m dq_m^{\mu}(v)$. Since

$$\frac{m}{n-m}B(m;n-1,\alpha)K^{\epsilon}(v_i) > B(m-1;n-1,\alpha)K^{\mu},$$

the effect on the seller's revenue is strictly positive. Therefore, $q_m^{\mu}(v) = 0$ for any v such that $v_i > v_+$ for some $i = 1, \ldots, n-m$. Further, since $K^{\epsilon}(w)$ is strictly increasing for $w > v_+$, we have $q_m^{\epsilon}(\rho_m^i(v)) = 1$ for $v_i = \max[v_1, \ldots, v_{n-m}]$. Finally, since

$$\frac{B(m;n-1,\alpha)}{n-m}K^{\epsilon}(w) \leq \frac{B(m-1;n-1,\alpha)}{m}K^{\mu}.$$

for all $w \leq v_+$, with equality if $w \in [v_-, v_+]$, if v is such that $\max[v_1, \ldots, v_{n-m}] \leq v_+$, there is a maximizer of the Lagrangian such that $q^\epsilon_m(\rho^i_m(v)) = 0$ whenever $v_i < v_-$, and $q^\epsilon_m(\rho^i_m(v)) = q^\mu_m(v)$ if $v_i \in [v_-, v_+]$.

For $m = 0$ and the first term in the Lagrangian, the strict concavity of $\pi(\cdot)$ implies $K^\epsilon(w)$ for $w < v_-$ crosses 0 at most once and only from below. Thus, for r that satisfies (3.9), it is point-wise maximizing to set $q^\epsilon_0(\rho^i_0(v)) = 1$ if $v_i = \max[v_1, \ldots, v_n]$ and $v_i > v_+$, or if $v_i = \max[v_1, \ldots, v_n]$ and $v_i \in [r, v_-)$; set $q^\epsilon_0(\rho^i_0(v)) = 1/k$ if $v_i \in [v_-, v_+]$, $\max[v_1, \ldots, v_n] \in [v_-, v_+]$ and $\#\{j : v_j \in [v_-, v_+]\} = k$; and set $q^\epsilon_0(\rho^i_0(v)) = 0$ otherwise.

For $m = n$ and the second term in the Lagrangian, we have $q^\mu_n = 1/n$ because $K^\mu > 0$. □

References

Mohammad Akbarpour and Shengwu Li. Credible auctions: A trilemma. *Econometrica*, 88(2):425–467, 2020. doi: 10.3982/ECTA15925. URL `https://onlinelibrary.wiley.com/doi/abs/10.3982/ECTA15925`.

Elchanan Ben-Porath, Eddie Dekel, and Barton L. Lipman. Optimal allocation with costly verification. *The American Economic Review*, 104(12):3779–3813, 2014. ISSN 00028282. URL `http://www.jstor.org/stable/43495357`.

Helmut Bester and Roland Strausz. Contracting with imperfect commitment and the revelation principle: The single agent case. *Econometrica*, 69(4):1077–1098, 2001. doi: 10.1111/1468-0262.00231. URL `https://onlinelibrary.wiley.com/doi/abs/10.1111/1468-0262.00231`.

G. Butters. Equilibrium distributions of sales and advertizing prices. *Review of Economic Studies*, 44:465–491, 1977.

R. Denekere and Sergei Severinov. Screening when some agents are nonstrategic: does a monopoly need to exclude? *Rand Journal of Economics*, 37(4):816–841, 2006.

Peter R. Dickson and Alan G. Sawyer. The price knowledge and search of supermarket shoppers. *Journal of Marketing*, 54(3):251–334, July 1990.

K. Hendricks and T. Wiseman. How to sell (or procure) in a sequential auction market*. 2020.

A. Kolotilin, H. Li, and W. Li. Optimal limited authority for principal. *Journal of Economic Theory*, 148(6):2344–2382, 2013.

Q. Liu, K. Mierendorff, and X. Shi. Auctions with limited commitment. Columbia University working paper, 2014.

Preston McAfee. Mechanism design by competing sellers. *Econometrica*, 61(6): 1281–1312, November 1993.

V. Skreta. Optimal auction design under non-commitment. *Journal of Economic Theory*, 159:854â–890, 2015.

D. Stahl. Oligopolistic pricing and advertising. *Journal of Economic Theory*, 64 (1):162–177, 1994.

H. Varian. A model of sales. *American Economic Review*, 70(4):651–659, 1980.

Gabor Virag. Competing auctions: Finite markets and convergence. *Theoretical Economics*, 5:241–274, 2010.