Model Secrecy and Stress Tests*

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Abstract

Conventional wisdom holds that the models used to stress test banks should be kept secret to prevent gaming. We show instead that secrecy can be suboptimal, because although it deters gaming, it may also deter socially desirable investment. When the regulator can choose the minimum standard for passing the test, we show that secrecy is suboptimal if the regulator is sufficiently uncertain regarding bank characteristics. When failing the bank is socially costly, then under some conditions, secrecy is suboptimal when the bank's private cost of failure is either sufficiently high or sufficiently low. Finally, we relate our results to several current and proposed stress testing policies.

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1 Introduction

According to Federal Reserve officials, the models that are used to stress test banks are kept secret to prevent banks from gaming them. Indeed, if a bank knows that the Fed's models underestimate the risks of some class of assets, the bank can invest in those assets without fear of failing the test. However, banks complain about this secrecy, claiming that even their best efforts to prepare for a test could result in unexpected and costly failure.[1]

Our main contribution is to present conditions under which, contrary to conventional wisdom and the statements of some policymakers, fully revealing the stress model to banks is optimal.[2] The results build on the idea that hidden models make banks cautious about risky investment, which could have two effects: banks may game less, but they may also invest less in socially desirable assets. Revealing the model leads to a better social outcome if the second effect dominates. This idea leads to three main results. First, if banks are sufficiently cautious about risky investment or if failing the test is sufficiently costly to them, revealing the regulator's model is optimal because it prevents underinvestment in socially desirable assets. Second, even if the regulator can adjust the test to make it easier to pass, revealing may still be optimal if uncertainty about the bank characteristics is sufficiently high, or if the regulator is forced to apply the same test to sufficiently different banks. Third, if there is some social cost when banks fail the test, then the optimal disclosure policy may be nonmonotonic in bank characteristics. For example, revealing could be optimal when the bank's bias toward risky investment or the bank's private cost of failure is either sufficiently high or sufficiently low.

In our baseline model, the bank can invest in one of two portfolios: a safe

[1] A recent proposal from the Federal Reserve suggests enhanced disclosure of the Fed models, such as revealing key variables and some equations, and illustrating how the Fed model will work on some hypothetical loan portfolios; but even under this proposal, the Fed will not reveal the exact models. See https://www.gpo.gov/fdsys/pkg/FR-2017-12-15/pdf/2017-26856.pdf

[2] See former Fed Governor Tarullo's speech for arguments against fully revealing the model. https://www.federalreserve.gov/newsevents/speech/tarullo20160926a.htm

portfolio, which will surely pass the test, or a risky portfolio, which may or may not pass the test. We assume that the bank always prefers to invest in the risky portfolio, whereas the regulator prefers the risky portfolio only if its value during a crisis is sufficiently high. This value is represented by the state of nature. The bank knows more than the regulator about the value of the risky portfolio, and for simplicity, we assume that the bank observes the state with certainty.

We capture the idea of a hidden stress test model by assuming that the regulator observes a noisy signal of the state, and that the regulator passes a bank that invested in the risky portfolio if and only if the signal realization is above some threshold. If the bank fails the test, the regulator forces the bank to alter its portfolio, which we assume is costly for the bank. If the bank passes the test, the regulator leaves the bank's portfolio unchanged. Because the regulator bases his decision on a noisy signal, he could err by passing a bank that invested in a socially undesirable portfolio or by failing a bank that invested in a socially desirable portfolio.

When the regulator's model is hidden, the bank fears failure and is therefore cautious, investing only when its privately observed state exceeds some threshold. We refer to this threshold as the bank's *cautious threshold*. In contrast, when the regulator reveals his signal, and that signal exceeds the passing threshold, the bank invests in the risky portfolio regardless of its privately observed state. So the bank may invest in the risky portfolio even if it knows that doing so is harmful to society. In other words, the bank may game the test.

We compare between two disclosure regimes: a transparent regime under which the regulator reveals his signal to the bank before the bank selects a portfolio, and a secrecy regime in which the regulator's signal is kept secret. We focus on two cases. In the first case, the regulator must follow an exogenously given threshold for passing or failing the bank.[3] In the second case,

[3]For example, the regulator must ensure that the bank's capital during an adverse stress scenario does not fall below some predetermined level.

the regulator can choose the passing threshold optimally. So in the second case, the regulator has two tools to influence the bank's portfolio decision: the disclosure regime and the standard to which the bank is held. In both cases, the regulator announces and commits to the passing threshold publicly before the bank selects its portfolio.

In the first case with an exogenously given threshold, we show that revealing is optimal if the bank's cautious threshold is sufficiently high. This happens, for example, if the bank's cost of failing the test is sufficiently high. Intuitively, in this case, the bank's fear of failing the test leads to a significant reduction in socially beneficial investment, and this reduction more than offsets the benefits from a reduction in a socially harmful investment.

In the second case in which the regulator can choose the passing threshold optimally, he can reassure an overly cautious bank by lowering the passing threshold, thereby making the test easier to pass. However, the bank's cautious threshold depends not only on the difficulty of the test but also on the bank's characteristics (e.g., cost of failing the test). If the regulator is certain about the bank characteristics, he can precisely calibrate the bank's cautious threshold by adjusting the passing threshold, so it is optimal to not reveal. However, precise calibration is impossible when bank characteristics are unknown. We show that, under some conditions, if the regulator is sufficiently uncertain about the bank's characteristics, then revealing is optimal.

Finally, we focus on another force that increases the benefit of revealing the regulator's model. Failing the test and the resulting change in the bank's portfolio might be costly not only for the bank but also for society. We show that if the social cost of failing the test is sufficiently high, it is optimal to reveal the regulator's signal. If instead, the social cost of failing the test is low, the optimal disclosure regime depends on the bank's cautious threshold, and in particular, on the bank's cost of failing the test. Interestingly, the relationship between the optimal disclosure policy and the bank's cost of failure is not necessarily monotone.

For example, under some conditions, revealing is optimal when the bank's private cost of failing the test is either sufficiently high or sufficiently low. Intuitively, if the cost is high, then fear of failing the test deters the bank from taking a socially desirable risk, and so it is optimal to reveal. If the cost is low, fear of failure does little to deter investment in socially harmful assets. But then it is better to reveal to avoid the social cost of failing the bank. In other words, in this case, providing incentives to the bank via model secrecy is too costly for the regulator.

We are currently working on understanding optimal disclosure when the regulator can commit to a more general disclosure rule as in the Bayesian persuasion literature (e.g., Gentzkow and Kamenica (2011)). Our preliminary results (Section 7) suggest that if no disclosure leads the bank to underinvest in socially desirable assets, it is optimal to reveal some of the signals that fall below the passing threshold and pool the rest. In this case, a simple cutoff rule is optimal, and as the bank's private cost of failure increases (i.e., as the bank becomes more cautious), it is optimal to reveal more information. However, if no disclosure leads the bank to overinvest in socially undesirable assets, the optimal disclosure rule is more complicated, and is, in general, nonmonotone. In the special case in which the regulator can observe only two signals, we show that under some conditions, the regulator discloses more information when the bank's private cost of failure decreases. This is consistent with the intuition we developed earlier that it is optimal to reveal more information when fear of failing a hidden test does only little to deter the bank from investing in socially harmful assets.

Before concluding, we discuss additional policy implications from our model. In particular, we relate our results to three specific policies: the current policy of giving banks a short time to revise their capital plans, the proposal to reveal the Fed's estimated losses on hypothetical portfolios, and the suggestion to accompany greater model transparency with increased capital requirements.

2 Related Literature

The existing literature has focused on disclosure of regulators' stress test *results* to *investors*.[4] In contrast, we focus on disclosure of regulators' stress test *models* to *banks*. To our knowledge, we are the first paper to study this problem.

Our setting is a principal-agent problem in which the principal (the regulator) and agent (the bank) each have private information, the agent takes an action, and the principal can take a follow up action, which is costly both to the agent and to the principal. Our focus is on whether the principal should reveal his private information before the agent takes the action. Levit (2016) also considers a setting in which a principal can reverse the agent's action. In his basic setting, the principal is more informed than the agent, so intervention can protect the agent from bad outcomes. His paper shows that in some cases the principal can obtain a better outcome by recommending an action to an agent and committing not to intervene. In our setting, however, intervention is bad for the agent and is crucial for providing incentives; instead, the principal chooses whether or not to disclose information related to his intervention policy.

As in the delegation literature initiated by Holmstrom (1985)[5], we rule out transfers between the two parties. The case in which the regulator reveals his information corresponds to a standard delegation problem in which the principal delegates partial authority to the agent. In particular, by revealing his signal, the regulator effectively restricts the bank's action space to those actions that will surely pass the test. In contrast, the case in which the regulator

[4]See Goldstein and Leitner (2017); Williams (2015); Goldstein and Sapra (2014); Bouvard, Chaigneau, and Motta (2015); Faria-e Castro, Martinez, and Philippon (2016); Inostroza and Pavan (2017); Orlov, Zryumov, and Skrzypacz (2017); Gick and Pausch (2012). Also Leitner and Yilmaz (forthcoming) study the extent to which the regulator should rely on or monitor bank internal risk models.

[5]See also Dessein (2002); Amador and Bagwell (2013, 2016); Amador, Bagwell, and Frankel (2017); Grenadier, Malenko, and Malenko (2016); Chakraborty and Yilmaz (2017); Harris and Raviv (2005, 2006); Halac and Yared (2016).

does not reveal his information is new to this literature and can be thought of as "delegation with hidden evaluation." The regulator does not restrict the set of actions that the bank can take (i.e., there is full delegation), but the regulator responds to the bank's action based on an evaluation process (a model) that is hidden from the bank. Our paper provides conditions under which hiding the evaluation process is preferred to revealing it.

Our results on general disclosure rule relate to the Bayesian persuasion literature. Our setting is an example of a persuasion game with one sender and one privately informed receiver.[6] Because of the unique structure of our problem, we cannot apply existing solution methods.

The idea that uncertainty regarding the regulator policy can affect incentives appears in other settings. For example, Lazear (2006) shows hidden tests could be a way to induce a socially optimal action, such as studying or not speeding. In his setting, the regulator knows what the socially optimal action is, whereas in our setting the regulator does not know. The possibility of wrongful punishment in our setting can create excessive caution in banks, which is the driving force behind our results. Freixas (2000) offers some justification for "constructive ambiguity" of bank bailout policy by showing that under some conditions, it is optimal for the regulator to use a mixed bailout strategy. In our paper, the regulator follows a deterministic policy rule to pass or fail a bank, but the rule is based on information that could be unknown to the bank.

Finally, there is a large empirical literature that documents how political and regulatory uncertainty can affect the real economy, including reducing investment.[7] In particular, Gissler, Oldfather, and Ruffino (2016) offer evidence which suggests that uncertainty about the regulation of qualified mortgages

[6]Gentzkow and Kamenica (2011); Kolotilin (2016); Kolotilin, Mylovanov, Zapechelnyuk, and Li (2017)

[7]For example, Julio and Yook (2012) document that high political uncertainty causes firms to reduce investment during election years. Fernández-Villaverde, Guerrón-Quintana, Kuester, and Rubio-Ramírez (2015) document that temporarily high uncertainty about fiscal policy reduces output, consumption, and investment. See also Pástor and Veronesi (2013) and Baker, Bloom, and Davis (2016).

caused banks to reduce mortgage lending. The literature is consistent with the idea in our paper that hidden tests could induce the bank to invest less.

3 Model

There is a bank and a regulator. The bank can invest in either a risky asset or a safe asset. The payoff from investing in the risky asset depends on an unobservable state ω, which represents the value of the risky asset in a crisis. The bank's payoff is $u(\omega)$ and the regulator's payoff is $v(\omega)$. These payoff functions take into account the probability of a crisis, the resulting losses, the payoffs during normal times, etc. The payoff from investing in the safe asset does not depend on the state, and is normalized to zero for both the bank and regulator. That is, u and v are the relative gains from investing in the risky asset, compared to the safe asset. The regulator's payoff represents the payoff to society. We assume that u and v are continuous and differentiable. We also assume that:

Assumption 1. *u and v are strictly increasing.*

Assumption 2. *For all $\omega \in \Omega$, $u(\omega) > 0$.*

Assumption 3. $v(\underline{\omega}) < 0 < v(\bar{\omega})$

Assumption 1 implies that both the regulator and the bank prefer higher value in a crisis. Assumption 2 implies that the bank always prefers to invest in the risky asset, and Assumption 3 implies that the regulator prefers the risky asset only if its value during a crisis is sufficiently high. These assumptions capture the conflict of interest between the bank and the regulator. For example, the bank may not internalize the social cost associated with risk. For use below, we define ω_r to be the unique zero of v; so the regulator prefers the risky asset if and only if $\omega \geq \omega_r$.[8]

[8] It is not crucial for our results that the bank prefers the risky asset in every state. The only thing that matters is that there are states in which the banks prefers the risky asset but the regulator does not.

After the bank chooses its portfolio, the regulator assesses the value of the portfolio during a crisis—i.e., the regulator performs a stress test. We assume that if the bank chooses the safe asset, the bank always passes the test. Hence, the only relevant assessment is that of the risky asset. This assessment is represented by a noisy signal s of ω, and it is assumed that if the bank chooses the risky asset, the bank passes the test if and only s is above some threshold s_p. We consider the case in which the passing threshold s_p is exogenous as well as the case in which the threshold is optimally chosen by the regulator.

In practice, the regulator's model for assessing the portfolio value is a complicated function of the stress scenario, regulator data, the bank's portfolio, and bank-specific data about asset performance. However, in our simplified setting, there is only one stress scenario, only one risky asset, and no communication from the bank to the regulator regarding its private information. So in our setting, revealing the model reduces to revealing the regulator's forecast s of the risky asset's value. As a result, hereafter we use the phrases "revealing the model" and "revealing s" interchangeably.

Passing the test means the regulator leaves the bank's portfolio unchanged, whereas failing the test means the regulator requires the bank to replace the risky asset with the safe asset. This replacement incurs a cost $c_b > 0$ to the bank and $c_s \geq 0$ to the regulator. For example, these costs could represent the opportunity cost of delaying investment in the safe asset[9]. The parameter c_b could also represent the loss of reputation to the bank's manager after failing the test.[10] The parameter c_s could capture the affect of altering the bank's portfolio on potential borrowers or other banks (e.g., due to fire sales), and it could also capture the idea that news of failure may cause a panic, leading to

[9] We assume that investment in the risky asset is available only before the stress test.

[10] In practice, there could be other consequences for failing the test. What's important for our model is that failing is costly for the bank. Moreover, assuming that c_b does not depend on the state is not crucial for our results. This assumption is used when we later show that the bank follows an investment threshold rule, but other specifications (e.g., c_b decreases in omega) will generate similar results.

contagion.

The state ω is drawn from a continuous cumulative distribution function $G(\omega)$ with support $\Omega \equiv [\underline{\omega}, \bar{\omega}] \subset \mathbb{R}$. Conditional on ω, the noisy signal s is drawn from a continuous cumulative distribution function $F(s|\omega)$ with density $f(s|\omega)$ and support $S = [\underline{s}, \bar{s}] \subset \mathbb{R}$.

Assumption 4 (Monotone Likelihood Ratio Property)**.** *If $\omega' > \omega$, then the ratio $f(s|\omega')/f(s|\omega)$ is strictly increasing in s.*

Assumption 4 implies that $1 - F(s|\omega)$ is strictly increasing in ω.[11] That is, the regulator is more likely to observe higher signals when the state ω is higher. Note, however, that the regulator could err. He could pass a bank that invested in a sociably undesirable asset or fail a bank that invested in a socially desirable asset.

The bank privately observes the state ω and the regulator privately observes the signal s. This captures the idea that the bank knows more than the regulator about the value of the risky asset, but that the bank does not know the regulator's forecast. Everything else is common knowledge.

The focus of the paper is whether the regulator should reveal his private signal s to the bank. We start with the case in which the regulator can either reveal or not reveal his signal, and in Section 7, we explore more general disclosure rules. The sequence of events is as follows

1. The regulator publicly commits to either reveal or not reveal his private signal.

2. Nature chooses the state ω. The bank privately observes ω, and the regulator privately observes the signal s.

3. In accordance with his prior commitment in step (1), the regulator either reveals or does not reveal his signal.

4. The bank chooses the risky asset or the safe asset.

[11] See Milgrom (1981).

5. The regulator conducts the test, passing or failing the bank.

6. Payoffs are realized.

   - If the bank invested in the safe asset, both the bank and regulator receive 0.
   - If the bank invested in the risky asset, then if the bank passes the test, the bank receives $u(\omega)$ and the regulator receives $v(\omega)$; if the bank fails the test, the bank receives $-c_b$ and the regulator receives $-c_s$.

Later, we refer to investment in the risky asset simply as "investing" and to investment in the safe asset as "not investing."

4 Exogenous passing threshold

We begin our analysis with the case in which the passing threshold is given exogenously. We denote the passing threshold by s_p. So a bank that invests in the risky asset passes the test if and only if $s \geq s_p$. To simplify the exposition, we focus on the case in which the social cost c_s of failing the bank is zero. In Section 6, we discuss the case in which $c_s > 0$, which will give us more results.

We first characterize the bank's investment decision. Then, we compare the regulator payoffs under the two regimes: revealing the signal and not revealing. Assume that if the bank is indifferent between investing and not investing, the bank invests.

If the regulator reveals his signal s to the bank, the bank invests if and only if it expects to pass the test—that is, the bank invests when $s \geq s_p$, irrespective of ω. This follows because the bank's payoff from investing and passing the test is positive for all states $\omega \in \Omega$, the payoff from not investing is zero, and the payoff from investing and failing the test is negative. So when the regulator reveals s, the bank uses its knowledge of the test to act in a way that improves its payoff, regardless of the impact on society; i.e., the bank games the test.

If, instead, the regulator does not reveal his signal, the bank's action depends only on the bank's private information, the state ω. Conditional on ω, the bank's expected payoff from investing is $[1 - F(s_p|\omega)]u(\omega) - F(s_p|\omega)c_b$. In particular, with probability $1 - F(s_p|\omega)$, the bank passes the test and obtains $u(\omega)$, and with probability $F(s_p|\omega)$, the bank fails the test and suffers a cost c_b. If the bank does not invest, its payoff is zero. Hence, the bank invests in state ω if and only if

$$[1 - F(s_p|\omega)]u(\omega) - F(s_p|\omega)c_b \geq 0. \tag{1}$$

Next, we show that the bank follows a threshold investment policy: it invests if and only if the state is sufficiently high. Specifically, if the left-hand side of (1) is negative for all $\omega \in \Omega$, the bank never invests. Otherwise, denote the lowest[12] $\omega \in \Omega$ that satisfies (1) by $\omega_b(s_p)$. Because the left-hand side is strictly increasing in ω, the bank invests if and only if $\omega \geq \omega_b(s_p)$. We refer to $\omega_b(s_p)$ as the bank's *cautious threshold*.

The cautious threshold can be found as follows. If the left-hand side is positive at $\underline{\omega}$, then $\omega_b(s_p) = \underline{\omega}$; otherwise, $\omega_b(s_p)$ is the unique zero of the left-hand side, and by the implicit function theorem must be strictly increasing in s_p. Intuitively, when the threshold for passing the test is higher, the bank is less likely to invest because it is more afraid of failing the test. Also, note that ω_b increases in c_b. Intuitively, when the cost of failure is higher, the bank acts more cautiously, investing in fewer states.

The next lemma summarizes the results above:

Lemma 1. 1. *When the regulator does not reveal his signal to the bank, the bank invests if and only if $\omega \geq \omega_b(s_p)$.*

2. *$\omega_b(s_p)$ is increasing in s_p.*

[12] This exists because the left-hand side is continuous.

Given the bank's equilibrium strategy, we compare the regulator's payoff under both regimes. If the regulator reveals his signal s to the bank, the bank invests if and only if $s \geq s_p$. So in state ω, the bank invests with probability $1 - F(s_p|\omega)$. Taking the expectation across all states gives the regulator's expected payoff under the revealing regime:

$$V_r(s_p) \equiv \int_{\omega \geq \underline{\omega}} [1 - F(s_p|\omega)]v(\omega)dG(\omega). \tag{2}$$

If the regulator does not reveal his signal, the bank invests if $\omega \geq \omega_b(s_p)$, and if the bank invests in state ω, the bank passes the test with probability $1 - F(s_p|\omega)$. Hence, the regulator's expected payoff is

$$V_n(s_p) \equiv \int_{\omega \geq \omega_b(s_p)} [1 - F(s_p|\omega)]v(\omega)dG(\omega). \tag{3}$$

Equations (2) and (3) show the effect of revealing the regulator's signal s: the bank invests for more states ω. That is, when the regulator reveals his signal, the bank invests for all $\omega \in \Omega$, but when the regulator does not reveal his signal, the bank invests only if $\omega \geq \omega_b(s_p)$.

It is optimal for the regulator to reveal his signal if $V_r(s_p) \geq V_n(s_p)$. Rearranging terms, we obtain that it is optimal to reveal if and only if

$$\int_{\underline{\omega}}^{\omega_b(s_p)} [1 - F(s_p|\omega)]v(\omega)dG(\omega) \geq 0. \tag{4}$$

Hence, whether it is optimal to reveal depends on whether the additional investment in states $[\underline{\omega}, \omega_b(s_p)]$ is socially beneficial. As we explain below, the net effect from this additional investment on the regulator's expected payoff can be either positive or negative.

Specifically, if $\omega_b(s_p) \leq \omega_r$, the left-hand side of (4) is negative, capturing the idea that revealing the signal causes the bank to invest in socially undesirable projects. On the other hand, if $\omega_b(s_p) > \omega_r$, the left-hand side can be written as the sum of two terms:

$$\int_{\underline{\omega}}^{\omega_r} [1 - F(s_p|\omega)]v(\omega)dG(\omega) \tag{5}$$

and

$$\int_{\omega_r}^{\omega_b(s_p)} [1 - F(s_p|\omega)]v(\omega)dG(\omega). \tag{6}$$

Expression (5) is negative and represents the cost of revealing the signal, as mentioned above. We refer to this as the overinvestment effect of revealing the signal. Expression (6) is positive and represents a benefit of revealing the signal, which is that the bank invests in more states in which it is socially desirable to do so. Specifically, if the regulator does not reveal the signal, the bank does not invest in states $\omega \in [\omega_r, \omega_b(s_p))$, which would have given a positive social payoff $v(\omega)$. Revealing the signal avoids this underinvestment effect.

The discussion above suggests that it is optimal to reveal only if the underinvestment effect (6) of not revealing the signal is sufficiently high or the overinvestment effect (5) of revealing the signal is sufficiently low. The next proposition formalizes this intuition.

Proposition 1. *Given a passing threshold s_p such that $V_r(s_p) > 0$, there exists $\bar{\omega}_I \in (\omega_r, \bar{\omega})$ such that the regulator prefers to:*

$$\begin{cases} \textit{not reveal} & \textit{if } \omega_b(s_p) \in (\underline{\omega}, \bar{\omega}_I) \\ \textit{reveal} & \textit{if } \omega_b(s_p) > \bar{\omega}_I \\ \textit{either} & \textit{if } \omega_b(s_p) \in \{\underline{\omega}, \bar{\omega}_I\}. \end{cases}$$

The Proposition shows that whether it is optimal for the regulator to reveal his signal depends on the bank's cautious threshold ω_b. When ω_b is sufficiently high, it is optimal to reveal because not revealing induces the bank to invest too little in socially desirable projects. In contrast, if the cautious threshold

ω_b is sufficiently low, but still above $\underline{\omega}$, it is optimal to not reveal because then the bank invests less in socially undesirable projects.

Using Proposition 1, we can derive comparative statics as to how the regulator's optimal disclosure policy changes when model parameters change. For example, consider ω_b. Observe that the cautious threshold ω_b is increasing in the bank's cost c_b of failing the test. Hence, we have the following.

Corollary 1. *Given a passing threshold s_p such that $V_r(s_p) > 0$, there exists $\bar{c_b} \in (0, \infty)$ such that the regulator prefers to:*

$$\begin{cases} \textit{not reveal} & \textit{if } c_b \in (0, \bar{c}_b) \\ \textit{reveal} & \textit{if } c_b > \bar{c}_b \\ \textit{either} & \textit{if } c_b \in \{0, \bar{c}_b\}. \end{cases}$$

In general, the bank's cautious threshold ω_b, which reflects the bank's reluctance to invest when the regulator's signal is hidden, depends not only on the bank's cost c_b of failing the test, but also on the bank's utility function u. Particular functional forms for u could include parameters that describe various other features, such as risk aversion or the extent of conflict of interest $u(\cdot)-v(\cdot)$ with the regulator. If such parameters have a monotonic relationship to ω_b, they would produce comparative statics similar to Corollary 1.

5 Regulator can choose passing threshold

In this section, we analyze the case in which the regulator can choose the passing threshold optimally together with the disclosure policy.[13] For example, a higher threshold could reflect the fact that the regulator wants banks to have more capital during a crisis. We focus on two cases. In the first case

[13] Both decisions are made in step 1 in the sequence of events, and the regulator publicly announces and commits to both.

the regulator can tailor the passing threshold to individual banks, whereas in the second case, the regulator must apply same threshold to everyone. Such a constraint may reflect constraints on what regulators can do in practice. Alternatively, it can arise when the regulator is uncertain about the bank characteristics–in our case, the private cost c_b.

We let s_r denote the passing threshold that the regulator sets if he plans to reveal the test and s_n denote the passing threshold that the regulator sets if he does not plan to reveal. So the regulator payoff is $V_r(s_r)$ if he chooses to reveal the test and $V_n(s_n)$ if he chooses not to reveal. To rule out trivial cases in which not revealing weakly dominates revealing for all parameter values, we assume that s_r is interior: $s_r \in (\underline{s}, \bar{s})$. Then $V_r(s_r) > 0$.

The next proposition shows that when the regulator can tailor the passing threshold to individual banks, not revealing is preferred.

Proposition 2. *If the regulator can tailor the passing threshold to individual banks and $c_s = 0$, then for all $c_b \geq 0$ it is weakly optimal to not reveal.*

Intuitively, suppose the regulator used the optimal revealing threshold under both disclosure regimes. If not revealing were worse than revealing, it must be due to underinvestment. But then, the regulator could simply reduce the not-revealing threshold to eliminate underinvestment without inducing overinvestment, and also pass the bank more frequently than under revealing.

In the remainder of this section, we show that the result above may not hold if the regulator cannot tailor the passing threshold to individual banks. In particular, if there are many types of banks, the regulator may not be able to eliminate underinvestment without increasing overinvestment, as he did in Proposition 2, because a test that eliminates underinvestment for one type of bank could increase overinvestment for other types. This intuition will imply that if the regulator is sufficiently uncertain regarding the bank's type, in a way we make more precise below, then under some conditions, revealing is preferred to not revealing.

Formally, we assume that the bank's private cost of failure c_b is a random variable which is known to the bank but not to the regulator. We let $H(c) = P(c_b \leq c)$ be the cumulative distribution function which describes the regulator's beliefs about the distribution of c_b. We capture increased uncertainty regarding bank's type by looking at a sequence of distributions H that becomes more uncertain, in the sense of q-quantile-preserving spread. (The case $q = 0.5$ corresponds to the median-preserving spread.)

Definition 1. *H_2 is a q-quantile-preserving spread of H_1 if*

(i) H_1 and H_2 have the same q-quantile $z_q \equiv \min\{z | H_i(z) \geq q\}$,

(ii) $H_2(t) \geq H_1(t)$ for all $t \leq z_q$, and

(iii) $H_2(t) \leq H_1(t)$ for all $t \geq z_q$.

Note that now the payoff from not revealing will depend on the distribution of types H. In particular, by (1), a bank with private cost c_b will invest if and only if $c_b \leq [F(s|\omega)^{-1} - 1]u(\omega)$. So, the probability that the bank invests in state ω is

$$H([F(s|\omega)^{-1} - 1]u(\omega)),$$

and the regulator's payoff under not revealing is

$$V_n(s) \equiv \int_\Omega [1 - F(s|\omega)] H([F(s|\omega)^{-1} - 1]u(\omega)) v(\omega) dG(\omega). \tag{7}$$

In the special case in which the regulator knows the bank's type (i.e., H has all of the mass on a particular c_b), H is a step function, and (7) reduces to (3).

The next proposition shows that if q is sufficiently high, then in the limit, revealing is preferred to not revealing.

Proposition 3. *If q is sufficiently high, then for any sequence $\{H_i\}_{i=1}^\infty$ of distribution functions satisfying*

(i) H_{i+1} is a q-quantile-preserving spread of H_i for all $i \in \mathbb{N}$,

(ii) $\lim_{i \to \infty} H_i(c_b) = q$ *for all* $c_b > 0$,

revealing is strictly preferred to not revealing for high enough i.

The intuition behind Proposition 3 is as follows. In the limit, a measure q of types have arbitrarily small c_b and therefore require a high probability of test failure to deter them from overinvesting, whereas a measure $1 - q$ of types have arbitrarily high c_b and therefore require only a low probability of test failure to deter them from overinvesting. Since the regulator can choose only one passing threshold s_n, he must choose which group to calibrate to. If the regulator calibrates to the reckless types (low c_b), he sets a very high threshold s_n. In this case, he fails the bank almost all the time, and so his expected payoff is close to zero, which is less than $V_r(s_r) > 0$. If instead the regulator calibrates to the cautious types (high c_b), he sets a very low threshold s_n. In this case, he passes the bank almost all the time, and so the reckless types are emboldened, investing in all ω. If the fraction q of reckless types is high, this also gives a low payoff to the regulator.

In the proof of Proposition 3, we give a more precise meaning for what sufficiently high means. Specifically, we show that the result holds if $q \geq \nu$, where ν satisfies

$$V_r(s_r) = \nu \max\{E[v(\omega)], 0\} + (1 - \nu) \int_{\omega_r} v(\omega) dG(\omega). \tag{8}$$

As we explain below, ν is a measure of the noise in the regulator signal, where $\nu = 1$ is pure noise and $\nu = 0$ is zero noise. In particular, the term $\max\{E[v(\omega)], 0\}$ in (8) is the payoff that the regulator would obtain if his signal were perfectly uninformative. In this case, the regulator would either ban investment completely (set $s_r = \bar{s}$) and obtain a payoff of zero or always pass the bank (set $s_r = \underline{s}$) and obtain $E[v(\omega)]$. The term $\int_{\omega_r} v(\omega) dG(\omega)$ is the payoff that the regulator would obtain if his signal were perfectly informative. In this case, the regulator could set the threshold s_r so that the bank invests and passes the test only when $\omega \geq \omega_r$. The regulator's payoff under not revealing $V_r(s_r)$ is a weighted average of these two extremes; ν captures the weight on the perfectly uninformative extreme. Note that the condition $q \geq \nu$ is only a sufficient condition.

The following example illustrates the result in Proposition 3 for $q = 1/2$, so that $\{H_i\}$ is a sequence of median-preserving spreads.

Example 1. *Suppose* $\Omega = [0, 1]$*,* $G(\omega) = \omega$ *(i.e., uniform),* $f(s|\omega) = 2[(1 - s)(1 - \omega) + s\omega]$*,* $u(\omega) = \omega + 0.5$*, and* $v(\omega) = \omega - 0.5$*. So investment is socially desirable when* $\omega > 0.5$ *and socially undesirable when* $\omega < 0.5$*. Assume that the distribution* H *over* c_b *is lognormal with parameters* $\mu = \ln 2$ *and various values of* σ*. This amounts to fixing the median of* H *at 2 and changing the variance. (Note that in this example,* $E[v(\omega)] = 0$*,* $\int_{\omega_r} v(\omega) dG(\omega) = 0.125$*,* $s_r = 0.5$*, and* $V_r(s_r) = 1/24$*. So,* $\nu = 1/3$*.)*

Figure 1 illustrates the density function of c_b *for several values of* σ*. The figure shows that when* σ *is low, most of the mass is concentrated at the median of the distribution. When* σ *is high, the distribution puts a high mass on very low types and very high types.*

Figure 2 illustrates the regulator's payoffs $V_r(s_r)$ *and* $V_n(s_n)$*, as a function of various degrees of uncertainty* σ *about* c_b*. The payoff from revealing* $V_r(s_r)$ *does not depend on the level of uncertainty, whereas the payoff from not revealing* $V_n(s_n)$ *is strictly decreasing in the level of uncertainty. For a very low level of uncertainty, not revealing is strictly optimal. For a very high level of uncertainty, revealing is strictly optimal.*

6 Social Cost of Failing the Bank

In this section, we analyze the case in which there is a social cost c_s of failing the bank: $c_s > 0$. This social cost does not affect the bank's investment decision. Hence, the regulator's payoff from revealing is as in Equation (2) in Section 4. However, the regulator's payoff from not revealing becomes

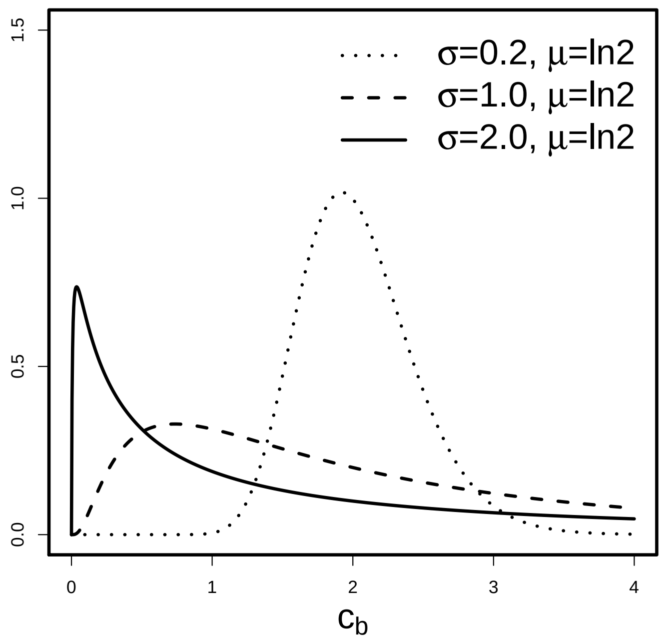

Figure 1: The distribution of c_b is lognormal, with fixed $\mu = \ln 2$, which implies a fixed median of 2. For fixed μ, the variance $[\exp(\sigma^2) - 1] \exp(2\mu + \sigma^2)$ is increasing in σ.

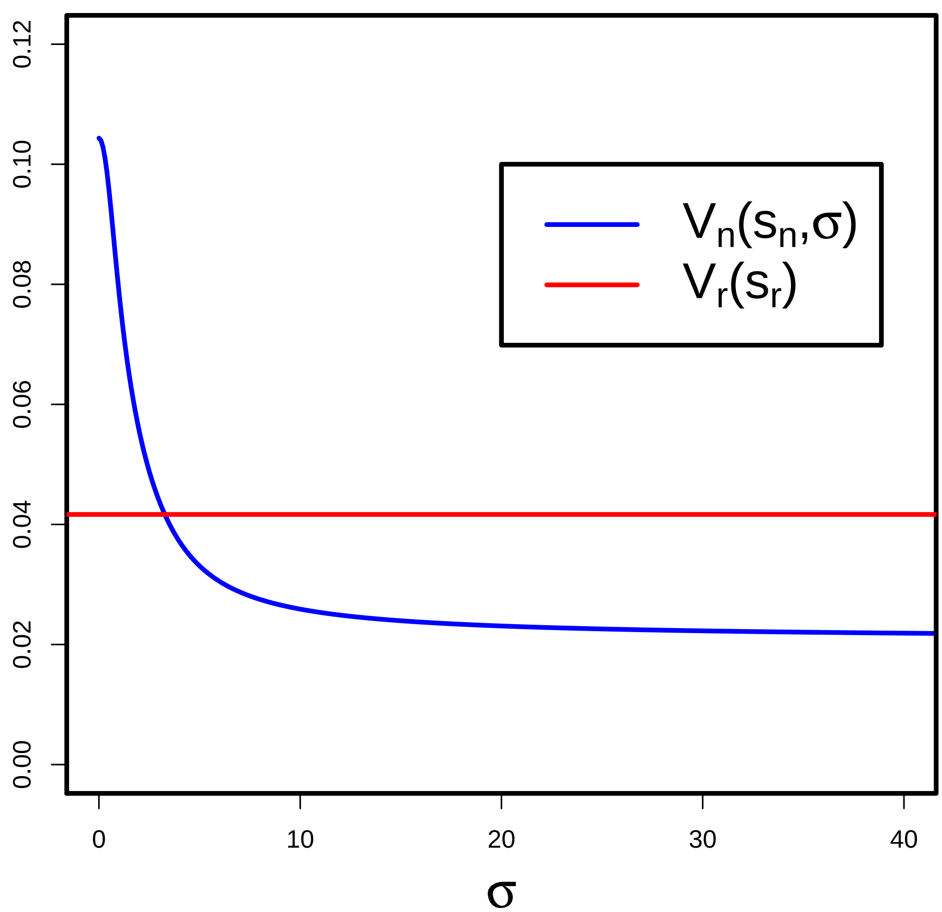

Figure 2: The regulator's payoff $V_n(s_n)$ from not revealing is decreasing in his uncertainty of the bank's type c_b.

$$V_n(s_p) \equiv \int_{\omega \geq \omega_b(s_p)} [1 - F(s_p|\omega)]v(\omega)dG(\omega) - c_s \int_{\omega \geq \omega_b(s_p)} F(s_p|\omega)dG(\omega). \quad (9)$$

The second term represents the additional social cost when the bank invests and fails the test.

We start with the case in which the passing threshold s_p is exogenous. In this case, it is optimal for the regulator to reveal his signal if $V_r(s_p) \geq V_n(s_p)$. Rearranging terms, we obtain that it is optimal to reveal if and only if

$$\int_{\underline{\omega}}^{\omega_b(s_p)} [1 - F(s_p|\omega)]v(\omega)dG(\omega) + c_s \int_{\omega \geq \omega_b(s_p)} F(s_p|\omega)dG(\omega) \geq 0. \quad (10)$$

Comparing (10) to (4), revealing the signal now has two effects on the regulator's payoff: not only does it cause the bank to invest in more states $[\underline{\omega}, \omega_b(s_p)]$, but it also avoids the social cost c_s of failing the bank. This suggests that if the social cost c_s of failing the bank is sufficiently high, it is optimal to reveal the signal. Otherwise, it is optimal to reveal only if the underinvestment effect (6) of not revealing the signal is sufficiently high or the overinvestment effect (5) of revealing the signal is sufficiently low. The next proposition formalizes this intuition.

Proposition 4. *Given a passing threshold s_p and regulator signal distribution $F(s|\omega)$, such that $V_r(s_p) > 0$, there exists a social cost of failure $\bar{c}_s > 0$ such that:*

1. *If $c_s > \bar{c}_s$, revealing is strictly preferred to not revealing.*
2. *If $c_s \leq \bar{c}_s$, then there exist $\underline{\omega}_I$, $\bar{\omega}_I \in \Omega$, with $\underline{\omega}_I \leq \bar{\omega}_I$ (with strict inequality if $c_s < \bar{c}_s$), such that:*
   - *(a) If $\omega_b(s_p) \in (\underline{\omega}_I, \bar{\omega}_I)$, not revealing is strictly preferred to revealing.*
   - *(b) If $\omega_b(s_p) < \underline{\omega}_I$ or $\omega_b(s_p) > \bar{\omega}_I$, revealing is strictly preferred to not revealing.*

(c) *If* $\omega_b(s_p) = \underline{\omega}_I$ *or* $\omega_b(s_p) = \bar{\omega}_I$*, the regulator is indifferent between revealing and not revealing.*

3. *As* c_s *decreases from* $\bar{c}_s$ *to 0,* $\underline{\omega}_I$ *strictly decreases to* $\underline{\omega}$ *and* $\bar{\omega}_I$ *strictly increases to a value strictly less than* $\bar{\omega}$*.*

Part 1 captures the idea that when the social cost of failure is high, the regulator would like to prevent failure by revealing the signal. In part 2, the social cost of failure is low, and whether revealing is optimal depends not only on the cost of failure but also on how not revealing affects the bank's investment decision. There are two circumstances in which it is optimal to reveal. The first case is when not revealing the signal does very little to prevent investment in bad projects—i.e., $\omega_b(s_p)$ is very low. In that case, revealing the signal induces only slightly worse investment behavior but avoids the cost of failure. The second case is when not revealing the signal deters not only bad investment but also much good investment—i.e., $\omega_b(s_p)$ is very high. In that case, revealing the signal permits this good investment and also avoids the cost of test failure.

Because $\omega_b(s_p)$ is increasing in c_b, we obtain the following immediate corollary, which is illustrated in Figure 3.

Corollary 2. *Given passing threshold* s_p*, if* $c_s > \bar{c}_s$*, revealing is optimal for all* $c_b \geq 0$*. If* $c_s \leq \bar{c}_s$*, there exist* $\underline{c}_b, \bar{c}_b \in \mathbb{R}_+$*, with* $\underline{c}_b \leq \bar{c}_b$ *(strict inequality if* $c_s < \bar{c}_s$*) such that the regulator's optimal policy is to*

$$\begin{cases} \textit{Reveal} & \text{if } c_b \in [0, \underline{c}_b) \cup (\bar{c}_b, \infty] \\ \textit{Not reveal} & \text{if } c_b \in (\underline{c}_b, \bar{c}_b) \\ \textit{Either} & \text{if } c_b \in \{\underline{c}_b, \bar{c}_b\}. \end{cases}$$

Furthermore, $\underline{c}_b$ *and* $\bar{c}_b$ *are respectively strictly increasing and strictly decreasing functions of* c_s*.*

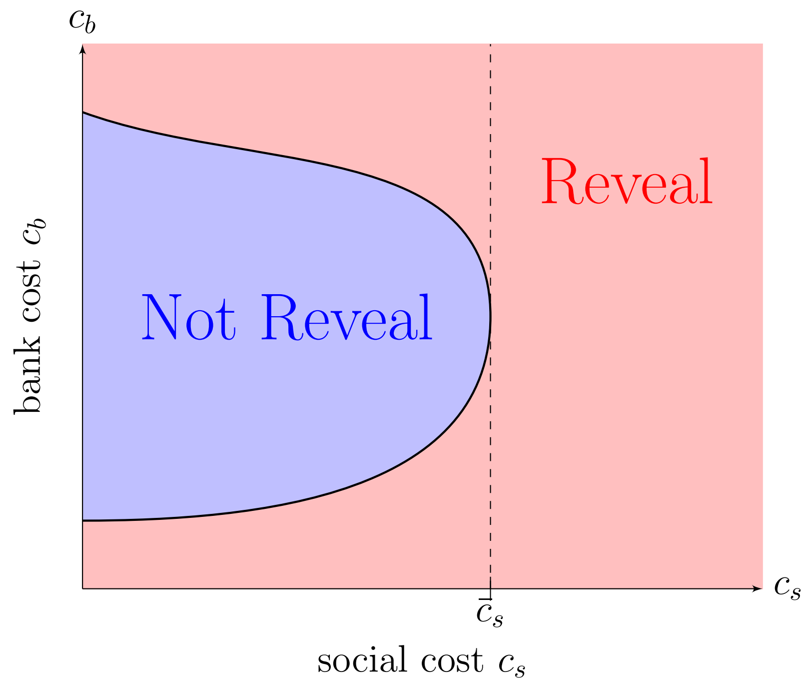

Figure 3: The regulator's optimal policy for the bank cost c_b and social cost c_s of stress test failure. For social cost $c_s < \bar{c}_s$, the optimal policy is nonmonotonic in the bank's private cost c_b of test failure.

The case in which the passing threshold is chosen optimally is analyzed in Proposition 5 below. Building on the intuition from Proposition 2, we show that for high enough c_b, not revealing is optimal because the optimal cautious threshold ω_b can be induced with a low passing threshold and therefore a low probability of costly test failure. However, following the intuition from Proposition 4, we show that for any positive c_s, revealing continues to be optimal when c_b is sufficiently low. In this case, the bank will reduce investment in socially undesirable projects only if the probability of failure is very high –i.e., the passing threshold is very high – but then the bank fails the test most of the time even if it invests in socially desirable projects.

Proposition 5. *If the regulator can choose the passing thresholds s_n and s_r, then for all $c_s \geq 0$, there exists a $\underline{c}_b > 0$ such that the regulator's optimal policy* *is to*

$$\begin{cases} \textit{Reveal} & \textit{if } c_b < \underline{c}_b \textit{ and } c_s > 0 \\ \textit{Not reveal} & \textit{if } c_b > \underline{c}_b \\ \textit{Either} & \textit{otherwise.} \end{cases}$$

Furthermore, as c_s increases to infinity, $\underline{c}_b$ strictly increases to infinity.

Interestingly, there are cases in which the optimal threshold under revealing is *lower* than under not revealing, but revealing is still preferred. This is in contrast to the view by some policy makers[14] that moving from a regime in which the regulator does not reveal his model to a regime in which the regulator reveals his model would need to be counteracted by an increase in minimum capital requirements (represented by the passing threshold in our model) to prevent gaming. Formally:

Proposition 6. *There exists a $c_s > 0$ and $c_b > 0$ such that revealing is strictly optimal and $s_r < s_n$.*

Intuitively, if the social cost of failure c_s is high enough, revealing is preferred to not revealing. If, in addition, the bank's private cost of failing the test c_b is low, deterring overinvestment under not revealing requires a very high threshold. That is, the threshold under not revealing is lower: $s_r < s_n$.

7 General disclosure

In this section, we analyze the case in which the regulator can choose a more general disclosure rule. We provide some preliminary results for the case in which the passing threshold is exogenous.

[14] See the departing speech by Fed Governor Daniel Tarullo: https://www.federalreserve.gov/newsevents/speech/tarullo20170404a.htm

7.1 Regulator's problem

Suppose the regulator can commit to a disclosure rule, which is defined by a set of messages M and a function that maps each signal $s \in S$ to a distribution over messages. We let $h(m|s)$ denote the probability of sending message $m \in M$ given signal s. To simplify the exposition, we assume that the sets M and S are finite.

Given $\{h(\cdot|s\}_{s \in S}$, a bank that observes state ω and message m expects to pass the test with probability

$$\Pr(s \geq s_p|\omega, m) = \frac{\sum_{s \geq s_p} f(s|\omega_i)h(m|s)}{\sum_s f(s|\omega_i)h(m|s)}, \tag{11}$$

where the equality follows from Bayes' rule. Hence, the bank's expected payoff from investing in the risky asset is

$$u(\omega)\Pr(s \geq s_p|\omega, m) - c_b \Pr(s < s_p|\omega, m). \tag{12}$$

By the MLRP property (Assumption 4), and since $u(\omega)$ is increasing in ω, it follows that the bank will follow a threshold strategy, investing if and only if the state ω is above some threshold ω_m. Hence, sending the message m is equivalent to recommending the investment threshold ω_m. Slightly abusing notation, $h(\omega_i|s)$ will denote the probability that the regulator recommends investment threshold ω_i upon observing signal s.

We denote the set of possible recommendations by $\Omega' \subset \Omega$ and assume that the recommendations $\underline{\omega}, \bar{\omega}, \omega_r$, and ω_b are includes in Ω'. Clearly, we must have

$$h(\omega_i|s) \geq 0 \quad \forall s \in S, \forall \omega_i \in \Omega' \tag{13}$$

$$\textstyle\sum_{\omega_i \in \Omega} h(\omega_i|s) = 1 \quad \forall s \in S \tag{14}$$

The bank will follow a recommendation ω_i if and only if, conditional on observing $\omega \geq \omega_i$, it cannot gain by not investing, and conditional on observing $\omega < \omega_i$, it cannot gain by investing. For an interior recommendation $\omega_i \in (\underline{\omega}, \bar{\omega})$, a necessary and sufficient condition for this is that the bank breaks even when he observes ω_i and obtains the recommendation ω_i:

$$u(\omega) \Pr(s \geq s_p | \omega, m) - c_b \Pr(s < s_p | \omega, m) = 0. \tag{15}$$

From (11), this reduces to the following *obedience constraint*:

$$u(\omega_i) \sum_{s \geq s_p} f(s|\omega_i|s) h(\omega_i|s) - c_b \sum_{s < s_p} f(s|\omega_i|s) h(\omega_i|s) = 0. \tag{16}$$

For the recommendations $\underline{\omega}$ and $\bar{\omega}$, the obedience constraints are similar but they will have weak inequalities:

$$u(\underline{\omega}) \sum_{s \geq s_p} f(s|\underline{\omega}|s) h(\underline{\omega}|s) - c_b \sum_{s < s_p} f(s|\underline{\omega}|s) h(\underline{\omega}|s) \geq 0. \tag{17}$$

$$u(\bar{\omega}) \sum_{s \geq s_p} f(s|\bar{\omega}|s) h(\bar{\omega}|s) - c_b \sum_{s < s_p} f(s|\bar{\omega}|s) h(\bar{\omega}|s) \leq 0. \tag{18}$$

Next, we calculate the regulator's expected payoff. Conditional on observing signal s, the regulator's payoff from recommending ω_i is

$$\begin{array}{ll} s \geq s_p: & v(\omega_i, s) \equiv \int_{\omega \geq \omega_i} v(\omega) f(\omega|s) d\omega \\ s < s_p: & c(\omega_i, s) \equiv -c_s \int_{\omega \geq \omega_i} f(\omega|s) d\omega \end{array} \tag{19}$$

The regulator's expected payoff is obtained by summing (19) across all signals and all possible recommendations and is given by:

$$\sum_{s \geq s_p} \sum_{\omega_i} f(s) v(\omega_i, s) h(\omega_i|s) - \sum_{s \geq s_p} \sum_{\omega_i} f(s) c(\omega_i, s) h(\omega_i|s) \tag{20}$$

The regulator's problem is to choose $\{h(\omega_i|s)\}_{\omega \in \Omega', s \in S}$ to maximize his expected payoff (20) such that (13), (14), and the obediences constraints (16)-

(18) hold. This is a linear programming problem. Since the feasible region is bounded and closed, and nonempty,[15] a solution exists.

We start with the case $c_s = 0$. Since $v(\omega_i, s)$ obtains a maximal value when $\omega_i = \omega_r$, an upper bound on the regulator's objective is $v(\omega_r, s) \sum_{s \geq s_p} f(s)$. We show below that this upper bound is achieved if $\omega_b \geq \omega_r$ but not if $\omega_b < \omega_r$.

7.2 Optimal disclosure when $\omega_b \geq \omega_r$

When $\omega_b \geq \omega_r$, no disclosure leads the bank to act too cautiously. (This corresponds to the no revealing case we analyzed earlier.) The next proposition shows that the regulator can overcome this problem via partial disclosure that pools all the passing signals $s \geq s_p$ with some of the failing signals $s < s_p$. We use $n_p = |\{s : s < s_p\}|$ to denotes the number of failing signals.

Proposition 7. *If $\omega_b \geq \omega_r$, then:*

1. There exist $\{q_s\}_{s<s_p} \in [0,1]^{n_p}$ such that
$u(\omega_r) \sum_{s \geq s_p} f(s|\omega_r) = c_b \sum_{s<s_p} f(s|\omega_r) q_s$.

2. For every $\{q_s\}_{s<s_p}$ that satisfies the condition in part 1, there is an optimal disclosure rule that gives the recommendations ω_r and $\bar{\omega}$, as follows:

$$h(\omega_r|s) = \begin{cases} 1 & \text{if } s \geq s_p \\ q_s & \text{if } s < s_p \end{cases} \tag{21}$$

$$h(\bar{\omega}|s) = \begin{cases} 0 & \text{if } s \geq s_p \\ 1 - q_s & \text{if } s < s_p \end{cases} \tag{22}$$

Intuitively, if $\omega_b > \omega_r$, the bank believes it has a very high probability of failing the test, and so it does not invest in some socially desirable states. The regulator would like to reassure the bank, inducing the investment threshold ω_r so that the bank invests in all socially valuable states. The way to do so is to recommend the threshold ω_r in all passing signals but few failing signals.

[15] In particular, no disclosure, namely, setting $h(\omega_b|s) = 1$ for every $s \in S$, and $h(\omega_i|s) = 0$ for every $\omega_i \neq \omega_b$ and $s \in S$, satisfies all the constraints.·

Then when the bank receives the recommendation ω_r, it knows the chance of a failing signal is low, so it is willing to follow the recommendation.

The next corollary shows that if $\omega_b \geq \omega_r$, optimal disclosure can be implemented via a simple cutoff rule that tells the bank whether the signal s is above or below some cutoff. If the signal is above s, the regulator recommends that the bank invest if and only if $\omega \geq \omega_r$. If the signal is below s, the regulator recommends that the bank does not invest.

Corollary 3. *1. If $\omega_b \geq \omega_r$, there exists a cutoff $\check{s} \leq s_p$ and an optimal disclosure rule that gives the recommendations ω_r and $\bar{\omega}$ such that*

$$h(\omega_r|s) = 1 \quad \text{if } s > \check{s}$$
$$h(\bar{\omega}|s) = 1 \quad \text{if } s < \check{s}$$

2. If $\omega_b = \omega_r$, then $\check{s} = \underline{s}$; that is, no disclosure is optimal. As ω_b increases (or equivalently as c_b increases), the cutoff $\check{s}$ increases; that is, the regulator discloses more information.

The first part captures the intuition that there are many ways to implement a perceived probability of failure that induces the bank to follow a given recommendation (ω_r in our case). One way is via a simple cut off rule.

The second part extends the intuition we developed earlier. Earlier, we showed that if the bank's private cost of failure is above some threshold, revealing is preferred to not revealing. With a general disclosure rule, full disclosure is optimal only in the limit when $c_b = \infty$, and the switch from no disclosure to full disclosure is gradual.

7.3 Optimal disclosure when $\omega_b < \omega_r$

When $\omega_b < \omega_r$, no disclosure induces the bank to act too recklessly, investing in socially undesirable states. In this case, the regulator would like to make the bank more cautious, inducing an investment threshold $\omega > \omega_b$. Extending the logic from the previous section, one could be tempted to try a disclosure rule in which the regulator recommends ω_r in all the failing signals and a few of the passing signals, so that when the bank receives the recommendation ω_r, it expects a sufficiently high probability of failure and is willing to follow the recommendation. However, in order to recommend ω_r in some passing signals, which increases the regulator's payoff relative to no disclosure, the regulator also needs to recommend some investment threshold $\omega < \omega_b$ in some other passing signals, and this decreases his payoff relative to his payoff under no disclosure. This suggests that optimal disclosure is more complicated in this case. As we show below, in general, a single cutoff rule will be suboptimal.

In fact, the next proposition shows that perhaps surprisingly, the recommended thresholds weakly increase for failing signals $s < s_p$ and weakly decrease for passing signals $s \geq s_p$. The proposition also shows that perhaps less surprisingly, recommended thresholds ω_i never exceed ω_r.

Proposition 8. *Under an optimal disclosure rule, the following hold:*

1. *For every* $\omega_i > \omega_r$ *and* $s > s_p$, $h(\omega_i|s) = 0$.
2. *For every* $\omega_i > \omega_j$ *and* $s < s' < s_p$, *if* $h(\omega_i|s) > 0$, *then* $h(\omega_j|s') = 0$.
3. *For every* $\omega_i < \omega_j \leq \omega_r$ *and* $s' > s \geq s_p$, *if* $h(\omega_i|s) > 0$, *then* $h(\omega_j|s') = 0$.

The intuition for the result that recommended thresholds weakly increase for failing signals (part 2 in the proposition) and weakly decrease for passing signals (part 3) is as follows. To induce $\omega_i \in (\underline{\omega}, \omega_r)$, the regulator must pool failing signals with passing signals. Since the purpose of disclosure is to make the bank more cautious, the most efficient way to do so is to put more mass in failing signals $s < s_p$ that ω_i thinks are relatively more likely and in passing signals $s \geq s_p$ that ω_i thinks are relatively less likely. By MLRP, higher types ω_i place more weight on higher signals s. This leads to increasing recommendations in failing signals $s < s_p$ and decreasing recommendations in passing signals $s \geq s_p$. For passing signals $s \geq s_p$, an additional force leads to decreasing recommendations ω_i. For higher passing signals $s \geq s_p$, the regulator is less worried about investment in low states ω (because by MLRP, these are less likely). So he can recommend a lower investment threshold ω_i.

7.3.1 Two signals

To gain more intuition, we illustrate the optimal disclosure rule for the case in which there are only two signals $s' < s_p \leq s$. For use below, let

$$\gamma(\omega, s, s') \equiv \frac{[v(\omega, s) - v(\underline{\omega}, s)]f(s)c_b f(s'|\omega)}{u(\omega)f(s|\omega)}, \tag{23}$$

and $\omega_{s,s'} \in \arg\max_{\omega \leq \omega_r} \gamma(\omega, s, s')$. To simplify the exposition, assume the maximizer $\omega_{s.s'}$ is unique.

Proposition 9. *Suppose $\omega_b < \omega_r$, there are only two signals $s' < s_p \leq s$, and $\omega_{s,s'} \geq \omega_b$. Then under the optimal disclosure rule:*

1. If the regulator observes s, he recommends ω', and if he observes s', he mixes between the recommendations $\omega_{s,s'}$ (probability q) and $\underline{\omega}$ (probability $1-q$), where

$$q = \frac{c_b f(s'|\omega_{s,s'})}{u(\omega_{s,s'})f(s|\omega_{s,s'})}.$$

2. As c_b falls, q falls. That is, the regulator discloses more information.

8 Policy Implications

As we discussed earlier, the insights from our model can shed light on current and suggested stress testing policies. In particular, our models suggests that revealing the test does not necessarily imply that minimum capital requirements need to increase (Proposition 6), and that revealing the test may be preferred if the regulator must apply the same test to every bank and bank are sufficiently different from one another (Section 5). Below we discuss more implications.

1. In practice, banks whose capital plans would lead their capital to fall below the required level are given a short time to adjust their plans.[16] In our model, this practice could imply a lower private cost for banks from failing the test.[17] Our model suggests that if the social cost of failing a bank is zero, a lower c_b implies the regulator should disclose less. However, if the social cost of a failing a bank is positive, the impact of a lower c_b on optimal disclosure is more nuanced. In particular, if c_b descends from a high value to a middle value, the regulator should disclose less. But if c_b descends from a middle value to a low value, the regulator should disclose more.

2. A widely expressed concern is that disclosing the Fed's models could increase correlations in asset holdings among banks subject to the stress tests (i.e., the largest banks), making the financial system more vulnerable to adverse financial shocks. A recent proposal suggests that the Fed reveals the outcome of applying its models to hypothetical loan portfolios.[18] An extension of our model would suggest that such enhanced disclosure could also increase correlations in asset holdings. The proposed hypothetical portfolios could serve as benchmark portfolios in which too many banks invest, leading to correlated investment. So just as in our basic model, in which the bank could underinvest in a socially valuable risky portfolio by choosing the safe portfolio for which the test results are predictable, in practice, banks could underinvest in their idiosyncratic risky portfolios, for which the test results are unpredictable, and overinvest in the benchmark risky portfolio, for which the test results are predictable.

[16] See https://www.gpo.gov/fdsys/pkg/FR-2017-12-15/pdf/2017-26856.pdf

[17] Earlier, we thought of c_b as the opportunity cost of delaying investment in the safe portfolio, but c_b could also represent other costs, such as the embarrassment involved with the public objections to a bank's capital plan.

[18] See https://www.gpo.gov/fdsys/pkg/FR-2017-12-15/pdf/2017-26856.pdf

9 Conclusion

We present conditions under which it is socially optimal for a regulator to reveal his stress testing model to the bank. The framework we present allows that banks may game a publicly known model, the chief concern underlying the Federal Reserve's policy of model secrecy. We show that despite the possibility of gaming, revealing the model may still be optimal, because uncertainty about the regulator's model may prevent banks from investing in socially valuable assets. In addition, even when the regulator can reassure cautious banks by relaxing the minimum standard to which they are held, revealing may be optimal if the regulator is sufficiently uncertain about bank characteristics. Finally, we show that if causing the bank to fail a stress test is socially costly, the optimal disclosure policy may be nonmonotonic in bank characteristics; that is, revealing may be optimal for banks that have very high or very low private cost of failure.

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Appendix

9.1 Proofs

Proof of Proposition 1. Let $J(\omega_b) = \int_{\underline{\omega}}^{\omega_b}[1 - F(s_p|\omega)]v(\omega)dG(\omega)$. By (4), revealing is strictly preferred if $J(\omega_b) > 0$, strictly not preferred if $J(\omega_b) < 0$, and the regulator is indifferent if $J(\omega_b) = 0$. Observe that $J'(\omega_b) = [1 - F(s_p|\omega_b)]v(\omega_b)dG(\omega)$, which has the same sign as $v(\omega_b)$. So by Assumptions 1 and 3, as ω_b increases from $\underline{\omega}$ to ω_r to $\bar{\omega}$, J strictly decreases from $J(\underline{\omega}) = 0$ to $J(\omega_r) < 0$ and then strictly increases to $J(\bar{\omega}) = V_r(s_p) > 0$. Let ω_I be the unique $\omega_b \in (\omega_r, \bar{\omega})$ for which $J(\omega_b) = 0$, and the proposition follows. □

Proof of Corollary 1. Note that $V_r(s_p) > 0$ implies $s_p < \bar{s}$. So given $\omega \in \Omega$, $1 - F(s_p|\omega) \in [0, 1)$, and because $c_b/(u(\omega) + c_b)$ is strictly increasing in c_b, there exists a unique $c_b(\omega) \in \mathbb{R}_+$ satisfying $1 - F(s_p|\omega) = c_b/(u(\omega) + c_b)$. Furthermore, $c_b(\omega)$ is strictly increasing in ω. Let $\bar{c}_b \equiv c_b(\bar{\omega}_I)$, and apply Proposition 1. □

Proof of Proposition 2. If $V_n(s_r) \geq V_r(s_r)$, then $V_n(s_n) \geq V_n(s_r) \geq V_r(s_r)$, so not revealing is weakly optimal. On the other hand, if $V_n(s_r) < V_r(s_r)$, then $\int_{\omega_b(s_r)}[1 - F(s_r|\omega]v(\omega)dG(\omega) < \int_{\underline{\omega}}[1 - F(s_r|\omega)]v(\omega)dG(\omega)$, which implies that $0 < \int_{\underline{\omega}}^{\omega_b(s_r)}[1 - F(s_r|\omega)]v(\omega)dG(\omega)$, so $\omega_b(s_r) > \omega_r$. If so, there exists $\hat{s} \in S$ such that $\hat{s} < s_r$ and $\omega_b(\hat{s}) = \omega_r$. Then $V_n(s_n) \geq V_n(\hat{s}) = \int_{\omega_b(\hat{s})}[1 - F(\hat{s}|\omega]v(\omega)dG(\omega) = \int_{\omega_r}[1-F(\hat{s}|\omega]v(\omega)dG(\omega) > \int_{\omega_b(s_r)}[1-F(s_r|\omega]v(\omega)dG(\omega) = V_r(s_r)$. □

Proof of Proposition 3. Because $\nu < q$, there exists $s_1 > \underline{s}$ such that for all $s < s_1$, all $i \in \mathbb{N}$, and all $\omega \in \Omega$, $1 - F(s|\omega) > \nu/q$ and $k_i(s, \omega) \in [q, 1]$. Also, there exists $s_2 < \bar{s}$ such that $s > s_2$ implies $\int_{\omega_r}[1-F(s|\omega)]v(\omega)dG(\omega) < V_r(s_r)$. Because $q < 1$, there exists $\epsilon > 0$ such that $\epsilon[\int_{\omega_r}^{\bar{\omega}}[1-F(s|\omega)]v(\omega)dG(\omega)-\int_{\underline{\omega}}^{\omega_r}[1-F(s|\omega)]v(\omega)dG(\omega)] < (1-q)V_r(s_r)$ for all $s \in [s_1, s_2]$. By pointwise convergence of $\{H_i\}_{i=1}^{\infty}$, there exists $N \in \mathbb{N}$ such that $i \geq N$ implies $|k_i(s, \omega) - q| < \epsilon$ for all $(s, \omega) \in [s_1, s_2] \times \Omega$. Suppose $i \geq N$. If $s < s_1$, then $1 - F(s|\omega) > \nu/q$ and

$k_i(s,\omega) \in [q,1]$ for all $\omega \in \Omega$, so

$$\begin{aligned}
V_n(s,H_i) &= \int_{\underline{\omega}}^{\omega_r} [1-F(s|\omega)]k_i(s,\omega)v(\omega)dG(\omega) + \int_{\omega_r}^{\bar{\omega}} [1-F(s|\omega)]k_i(s,\omega)v(\omega)dG(\omega) \\
&< \int_{\underline{\omega}}^{\omega_r} \frac{\nu}{q} \cdot q \cdot v(\omega)dG(\omega) + \int_{\omega_r}^{\bar{\omega}} 1 \cdot 1 \cdot v(\omega)dG(\omega) \\
&= \nu \int_{\underline{\omega}}^{\omega_r} v(\omega)dG(\omega) + (\nu + 1 - \nu) \int_{\omega_r}^{\bar{\omega}} v(\omega)dG(\omega) \\
&= \nu \int_{\underline{\omega}}^{\bar{\omega}} v(\omega)dG(\omega) + (1-\nu) \int_{\omega_r}^{\bar{\omega}} v(\omega)dG(\omega) \\
&\le \nu \max\{E[v(\omega)], 0\} + (1-\nu) \int_{\omega_r}^{\bar{\omega}} v(\omega)dG(\omega) = V_r(s_r).
\end{aligned}$$

If $s \in [s_1, s_2]$, then

$$\begin{aligned}
V_n(s,H_i) &= \int_{\underline{\omega}}^{\omega_r} [1-F(s|\omega)]k_i(s,\omega)v(\omega)dG(\omega) + \int_{\omega_r}^{\bar{\omega}} [1-F(s|\omega)]k_i(s,\omega)v(\omega)dG(\omega) \\
&< \int_{\underline{\omega}}^{\omega_r} [1-F(s|\omega)](q-\epsilon)v(\omega)dG(\omega) + \int_{\omega_r}^{\bar{\omega}} [1-F(s|\omega)](q+\epsilon)v(\omega)dG(\omega) \\
&= qV_r(s) + \epsilon \left(\int_{\omega_r}^{\bar{\omega}} [1-F(s|\omega)]v(\omega)dG(\omega) - \int_{\underline{\omega}}^{\omega_r} [1-F(s|\omega)]v(\omega)dG(\omega) \right) \\
&< qV_r(s) + (1-q)V_r(s_r) \\
&\le qV_r(s_r) + (1-q)V_r(s_r) = V_r(s_r).
\end{aligned}$$

If $s > s_2$, then

$$\begin{aligned}
V_n(s,H_i) &= \int_{\underline{\omega}}^{\bar{\omega}} [1-F(s|\omega)]k_i(s,\omega)v(\omega)dG(\omega) \\
&\le \int_{\omega_r}^{\bar{\omega}} [1-F(s|\omega)]k_i(s,\omega)v(\omega)dG(\omega) \le \int_{\omega_r}^{\bar{\omega}} [1-F(s|\omega)]v(\omega)dG(\omega) < V_r(s_r).
\end{aligned}$$

So for all $i \ge N$, $V_n(s_n, H_i) < V_r(s_r)$. $\square$

Proof of Proposition 4. For fixed s_p and $F(s|\omega)$, denote

$$J(c_s, \omega_b) = c_s \int_{\omega \geq \omega_b} F(s_p|\omega) dG(\omega) + \int_{\underline{\omega}}^{\omega_b} [1 - F(s_p|\omega)] v(\omega) dG(\omega).$$

By Equation (10), revealing is strictly preferred if $J(c_s, \omega_b(s_p)) > 0$, not revealing is strictly preferred if $J(c_s, \omega_b(s_p)) < 0$, and the regulator is indifferent if $J(c_s, \omega_b(s_p)) = 0$. Note that

$$\frac{\partial J(c_s, \omega_b)}{\partial \omega_b} = \Big([1 - F(s_p|\omega_b)] v(\omega_b) - c_s F(s_p|\omega_b) \Big) dG(\omega_b),$$

which has the same sign as

$$[1 - F(s_p|\omega_b)] v(\omega_b) - c_s F(s_p|\omega_b), \tag{24}$$

(Part 1): We first show that $J(c_s, \cdot)$ is strictly quasiconvex and, therefore, has a unique minimizer in $[\underline{\omega}, \bar{\omega}]$. If $c_s = 0$, then by Assumption 1, (24) is strictly decreasing when $\omega < \omega_r$ and strictly increasing when $\omega > \omega_r$. (By Assumption 4, $F(s_p|\omega) > 0$ for every $\omega > \underline{\omega}$.) If $c_s > 0$, then if $\omega_b \leq \omega_r$, (24) is strictly negative, and if $\omega_b > \omega_r$, (24) is strictly increasing in ω_b, crossing zero at most once. So $J(c_s, \cdot)$ is strictly quasiconvex. We denote the unique minimizer in $[\underline{\omega}, \bar{\omega}]$, as $\omega_m(c_s)$.

If (24) is negative for all $\omega \in \Omega$, then $\omega_m(c_s) = \bar{\omega}$. Otherwise, $\omega_m(c_s)$ is the unique zero of (24), and so, $\omega_m(c_s) \geq \omega_r$. Next, we show that the minimum $J(c_s, \omega_m(c_s))$ is increasing in c_s, and that there is a unique $c_s > 0$ such that $J(c_s, \omega_m(c_s)) = 0$. By the envelope theorem,

$$\frac{dJ(c_s, \omega_m(c_s))}{dc_s} = \frac{\partial J(c_s, \omega_m(c_s))}{\partial c_s} = \int_{\omega \geq \omega_m(c_s)} F(s_p|\omega) dG(\omega) \geq 0,$$

and the inequality is strict whenever $\omega_m(c_s) < \bar{\omega}$ Consider the value of $J(c_s, \omega_m(c_s))$ at the extreme point $c_s = 0$. If $c_s = 0$, (24) reduces to $[1 - F(s_p|\omega)] v(\omega)$ and therefore $\omega_m(0) = \omega_r$, so $J(0, \omega_m(0)) = J(0, \omega_r) < 0$. Now consider the value of

$J(c_s, \omega_m(c_s))$ as $c_s \to \infty$. For $\omega_m(c_s) \in [\omega_r, \bar{\omega})$, applying the implicit function theorem to (24) gives

$$\omega_m'(c_s) = \frac{F(s_p|\omega)}{[1 - F(s_p|\omega)]v'(\omega) + (c_s + v(\omega))(\partial[1 - F(s_p|\omega)]/\partial\omega)}\bigg|_{\omega=\omega_m(c_s)} \geq 0,$$

so $\omega_b(c_s)$ is weakly increasing in c_s, and must therefore converge to some limit $L \in [\underline{\omega}, \bar{\omega}]$ as $c_s \to \infty$. There are two cases. Case 1: $L = \bar{\omega}$. Then since $V_r(s_p) > 0$, it must be that $\lim_{c_s\to\infty} J(c_s, \omega_m(c_s)) \geq V_r(s_p) > 0$. Case 2: $L < \bar{\omega}$. Then $\lim_{c_s\to\infty} \frac{dJ(c_s,\omega_m(c_s))}{dc_s} = \int_{\omega\geq L} F(s_p|\omega)dG(\omega) > 0$, so $\lim_{c_s\to\infty} J(c_s, \omega_m(c_s)) = \infty$.[19] In either case, $J(c_s, \omega_m(c_s)) > 0$ for high enough c_s. By the intermediate value theorem, there exists a $c_s > 0$ for which $J(c_s, \omega_m(c_s)) = 0$. To show uniqueness, suppose there exist two zeros c_s' and c_s'', with $c_s' < c_s''$. That $J(c_s, \omega_m(c_s))$ is weakly increasing gives $dJ(c_s, \omega_m(c_s))/dc_s = 0$ for all $c_s \in [c_s', c_s'']$, which implies $\omega_m(c_s) = \bar{\omega}$, and therefore $J(c_s, \omega_m(c_s)) = J(c_s, \bar{\omega}) = V_r(s_p) > 0$, a contradiction. Denote by $\bar{c}_s$ the unique zero of $J(c_s, \omega_m(c_s))$. We have that $c_s > \bar{c}_s$ if and only if $J(c_s, \omega_m(c_s)) > 0$. So $c_s > \bar{c}_s$ if and only if for all $\omega_b(s_p) \in \Omega$, $J(c_s, \omega_b(s_p)) \geq \min_{\omega\in\Omega} J(c_s, \omega) = J(c_s, \omega_m(c_s)) > 0$, and Part 1 is proved.

(Part 2): If $c_s < \bar{c}_s$, then by the proof of Part 1, $J(c_s, \omega_m(c_s)) < 0$. As ω increases from $\underline{\omega}$ to $\bar{\omega}$, the strict quasiconvexity of $J(c_s, \cdot)$ implies that $J(c_s, \cdot)$ strictly decreases from $J(c_s, \underline{\omega}) \geq 0$ to $J(c_s, \omega_m(c_s)) < 0$ and then strictly increases to $J(c_s, \bar{\omega}) > 0$. So there exist exactly two zeros of $J(c_s, \cdot)$ in Ω: a unique $\underline{\omega}_I \in [\underline{\omega}, \omega_m(c_s))$ and a unique $\bar{\omega}_I \in (\omega_m(c_s), \bar{\omega})$ which satisfy

$$J(c_s, \omega_b(s_p)) \begin{cases} < 0 & \omega_b(s_p) \in (\underline{\omega}_I, \bar{\omega}_I) \\ > 0 & \omega_b(s_p) \in [\underline{\omega}, \underline{\omega}_I) \cup (\bar{\omega}_I, \bar{\omega}] \\ = 0 & \omega_b(s_p) \in \{\underline{\omega}_I, \bar{\omega}_I\}. \end{cases}$$

Finally, if $c_s = \bar{c}_s$, then $J(c_s, \omega_m(c_s)) = 0$. The value $\omega_m(c_s)$ is the unique

[19] If $\lim_{x\to\infty} h'(x) > 0$, then $\lim_{x\to\infty} h(x) = \infty$.

minimizer of $J(c_s, \cdot)$, so $\omega_m(c_s)$ is the only zero of $J(c_s, \cdot)$, and for all $\omega_b(s_p) \neq \omega_m(c_s)$, $J(c_s, \omega_b(s_p)) > 0$.

(Part 3): From above, $\underline{\omega}_I$ and $\bar{\omega}_I$ satisfy $0 = J(c_s, \underline{\omega}_I(c_s)) = J(c_s, \bar{\omega}_I(c_s))$. Applying the implicit function theorem gives $\underline{\omega}_I'(c_s) = -\frac{\partial J}{\partial c_s} / \frac{\partial J}{\partial \omega_b}\Big|_{(c_s, \underline{\omega}_I(c_s))}$. Because $\omega_m(c_s)$ is the unique zero of (24) and $\underline{\omega}_I(c_s) < \omega_m(c_s)$, it must be that $\frac{\partial J}{\partial c_s} < 0$ at $(c_s, \underline{\omega}_I(c_s))$. Furthermore, $\frac{\partial J}{\partial \omega_b}$ is strictly positive unless $\underline{\omega}_I = \underline{\omega}$, which occurs only if $c_s = 0$. So $\underline{\omega}_I'(c_s) < 0$ for all $c_s \in (0, \bar{c}_s)$. Similarly, $\bar{\omega}_I'(c_s) = -\frac{\partial J}{\partial c_s} / \frac{\partial J}{\partial \omega_b}\Big|_{(c_s, \bar{\omega}_I(c_s))}$. Because $\bar{\omega}_I(c_s) > \omega_m(c_s) \geq \omega_r > \underline{\omega}$, it must be that $\frac{\partial J}{\partial c_s} > 0$ and $\frac{\partial J}{\partial \omega_b} > 0$ at $(c_s, \underline{\omega}_I(c_s))$, so $\bar{\omega}_I'(c_s) > 0$ for all $c_s \in [0, \bar{c}_s)$. Finally, if $c_s = 0$ and $\bar{\omega}_I = \bar{\omega}$, then $J = V_r(s_p) > 0$, a contradiction. So if $c_s = 0$, $\bar{\omega}_I < \bar{\omega}$. □

Proof of Corollary 2. Note that $V_r(s_p) > 0$ implies $s_p < \bar{s}$. So given $\omega \in \Omega$, $1 - F(s_p|\omega) \in [0, 1)$, and because $c_b/(u(\omega) + c_b)$ is strictly increasing in c_b, there exists a unique $c_b(\omega) \in \mathbb{R}_+$ satisfying $1 - F(s_p|\omega) = c_b/(u(\omega) + c_b)$. Furthermore, $c_b(\omega)$ is strictly increasing in ω. Let $\underline{c}_b \equiv c_b(\underline{\omega}_I)$ and $\bar{c}_b \equiv c_b(\bar{\omega}_I)$, and apply Proposition 4. □

Proof of Proposition 5. Since $s_r \in (\underline{s}, \bar{s})$, it follows that $V_r(s_r) > V_r(\bar{s}) = 0$. Fix some $c_s > 0$. Suppose $c_b = 0$. Then for all $s \in S$, $\omega_b(s, 0) = \underline{\omega}$. If $s = \underline{s}$, then $F(s|\omega) = 0$, so $V_n(c_s, 0, s) = \int_{\underline{\omega}} v(\omega) dG(\omega) = V_r(\underline{s}) < V_r(s_r)$. If $s > \underline{s}$, then $F(s|\omega) > 0$, so $V_n(c_s, 0, s) < V_r(s) \leq V(s_r)$. So for all $s \in S$, $V_n(c_s, 0, s) < V_r(s_r)$, and therefore $V_n(c_s, 0, s_n) < V_r(s_r)$. Next, we show that there exists a $c_b > 0$ such that V_n strictly dominates V_r. Note that $\int_{\omega_r} v(\omega) dG(\omega) > V_r(s_r)$. By the continuity of $F(s|\omega)$, there exists an $s \in (\underline{s}, \bar{s})$ such that $\int_{\omega_r} ([1 - F(s|\omega)] v(\omega) - F(s|\omega) c_s) dG(\omega) > V_r(s_r)$. Because s is interior, the image of $(0, \infty)$ under $\omega_b(s, \cdot)$ is $[\underline{\omega}, \bar{\omega}]$, so there exists a $c_b > 0$ such that $\omega_b(s, c_b) = \omega_r$. Therefore, $V_n(c_s, c_b, s_n) \geq V_n(c_s, c_b, s) > V_r(s_r)$. By the continuity of $V_n(c_s, \cdot, s_n(c_s, \cdot))$, there exists at least one $\underline{c}_b > 0$ such that $V_n(c_s, \underline{c}_b, s_n(c_s, \underline{c}_b)) = V_r(s_r)$. We now show that $V_n(c_s, \cdot, s_n(c_s, \cdot))$ is strictly increasing at any $\underline{c}_b$, which implies $\underline{c}_b$ is unique. By the envelope theorem,

$$\left.\frac{dV_n(c_s, c_b, s_n(c_s, c_b))}{dc_b}\right|_{\underline{c}_b} = \left.\frac{\partial V_n(c_s, c_b, s_n(c_s, c_b))}{\partial c_b}\right|_{\underline{c}_b}$$

$$= -\frac{\partial \omega_b(s_n, c_b)}{\partial c_b}\Big([1 - F(s_n|\omega_b)]v(\omega_b) - F(s_n|\omega_b)c_s\Big)dG(\omega_b)\Big|_{\underline{c}_b}, \tag{25}$$

where we have abbreviated $s_n(c_s, c_b)$ and $\omega_b(s_n(c_s, c_b), c_b)$ with s_n and ω_b, respectively. If $\omega_b \in \{\underline{\omega}, \bar{\omega}\}$, then $V_n(c_s, \underline{c}_b, s_n(c_s, \underline{c}_b)) < V_r(s_r)$, a contradiction, so $\omega_b \in (\underline{\omega}, \bar{\omega})$, and therefore $\frac{\partial \omega_b(s_n, c_b)}{\partial c_b}\big|_{\underline{c}_b} > 0$. To sign the second factor of (25), consider

$$\left.\frac{\partial V_n(c_s, c_b, s)}{\partial s}\right|_{(\underline{c}_b, s_n)} = -\int_{\omega_b(s, c_b)} f(s|\omega)(v(\omega) + c_s)dG(\omega)$$

$$- \frac{\partial \omega_b(s, c_b)}{\partial s}\Big([1 - F(s_n|\omega_b)]v(\omega_b) - F(s_n|\omega_b)c_s\Big)dG(\omega_b)\Big|_{(\underline{c}_b, s_n)}. \tag{26}$$

Note that because $\omega_b(s_n(c_s, \underline{c}_b), \underline{c}_b) \in (\underline{\omega}, \bar{\omega})$, it must be that $\frac{\partial \omega_b(s, c_b)}{\partial s}\big|_{(s_n, \underline{c}_b)} > 0$ and $s_n(c_s, \underline{c}_b) \in (\underline{s}, \bar{s})$. If $[1 - F(s|\omega_b)]v(\omega_b) - F(s|\omega_b)c_s\big|_{(\underline{c}_b, s_n)} \geq 0$, then $v(\omega_b) \geq 0$, so $\omega_b(s_n(c_s, \underline{c}_b), \underline{c}_b) \in [\omega_r, \bar{\omega})$, which implies the first term of (26) is strictly negative, and the second term is weakly negative. But then $\frac{\partial V_n(c_s, c_b, s)}{\partial s}\big|_{(\underline{c}_b, s_n)} < 0$, contradicting the optimality of $s_n \in (\underline{s}, \bar{s})$. So $\big([1 - F(s|\omega_b)]v(\omega_b) - F(s|\omega_b)c_s\big)\big|_{(\underline{c}_b, s_n)} < 0$, and therefore from (25) we have that $\frac{dV_n(c_s, c_b, s_n(c_s, c_b))}{dc_b}\big|_{\underline{c}_b} > 0$. So wherever $V_n(c_s, c_b, s_n) = V_r(s_r)$, V_n must be strictly increasing in c_b; together with the continuity of $V_n(c_s, \cdot, s_n)$, this implies that $V_n(c_s, \cdot, s_n)$ must cross $V_r(s_r)$ exactly once, namely at $\underline{c}_b$. Suppose $c_s = 0$. If $c_b = 0$, then for all $s \in S$, $\omega_b(s, 0) = \underline{\omega}$, $V(0, 0, s) = V_r(s)$, so $s_n = s_r$, and therefore $V_n(s_n) = V_r(s_r)$. Using an identical argument to the case of $c_s > 0$, there exists a $c_b > 0$ such that V_n strictly dominates V_r. Now consider (25) for $c_s = 0$ and $c_b > 0$. First note that

$\omega_b < \bar{\omega}$ and therefore $s_n < \bar{s}$, as otherwise $V_n = 0$, which is strictly dominated by selecting some $s < \bar{s}$ such that $\omega_b(s, c_b) = \omega_r$. Next, note that if $[1 - F(s_n|\omega_b)]v(\omega_b) \geq 0$, then $\omega_b \in [\omega_r, \bar{\omega})$ and $s_n \in (\underline{s}, \bar{s})$, which implies the first term of (26) is strictly negative and the second term is weakly negative, contradicting the optimality of $s_n > \underline{s}$. So $([1 - F(s_n)]|v(\omega_b) < 0$, and since $\frac{\partial \omega_b(s,c_b)}{\partial c_b} \geq 0$, (25) implies $V_n(0, c_b, s_n(0, c_b))$ is weakly increasing in c_b. So for $c_s = 0$, let $\underline{c}_b$ be the smallest c_b such that $V_n(0, c_b, s_n(0, c_b)) = V_r(s_r)$. To show that for $c_s = 0$, $\underline{c}_b > 0$, we show that there exists a $c_b > 0$ such that $V_n(0, c_b, s_n) = V_r(s_r)$. Let $\hat{s}(c_b)$ be the highest $s \in S$ such that $\omega_b(s, c_b) = \underline{\omega}$. Then given $c_b > 0$, $s \leq \hat{s}(c_b)$ implies $\omega_b(s, c_b) = \underline{\omega}$, so $V_n(0, c_b, s) = V_r(s)$. Note that $\lim_{c_b \to 0} \hat{s}(c_b) = \bar{s}$, and $\lim_{\hat{s} \to \bar{s}} V_r(\hat{s}) - \int_{\underline{\omega}}^{\omega_r} [1 - F(\hat{s}|\omega)]v(\omega)dG(\omega) = 0$. So there exists a $\delta > 0$ such that $c_b < \delta$ implies $\hat{s} > s_r$ and $V_r(\hat{s}) - \int_{\underline{\omega}}^{\omega_r} [1 - F(\hat{s}|\omega)]v(\omega)dG(\omega) < V_r(s_r)$. Given $c_b < \delta$, $\max_{s \leq \hat{s}(c_b)} V_n(0, c_b, s) = \max_{s \leq \hat{s}(c_b)} V_r(s) = V_r(s_r)$, whereas $\max_{s > \hat{s}(c_b)} V_n(0, c_b, s) = \max_{s > \hat{s}(c_b)} V_r(s) - \int_{\underline{\omega}}^{\omega_b(s,c_b)} [1 - F(s|\omega)]v(\omega)dG(\omega) \leq V_r(\hat{s}) - \int_{\underline{\omega}}^{\omega_r} [1 - F(\hat{s}|\omega)]v(\omega)dG(\omega) < V_r(s_r)$. So for all $c_b < \delta$, $V_n(0, c_b, s_n(0, c_b)) = \max_{s \in S} V_n(0, c_b, s) = V_r(s_r)$. Therefore, $\underline{c}_b \geq \delta > 0$. To show that $\underline{c}_b$ is a strictly increasing function of c_s, apply the implicit function theorem to $V_r(s_r) = V_n(c_s, \underline{c}_b(c_s), s_n(c_s, \underline{c}_b(c_s)))$ and the envelope theorem to get

$$0 = \frac{dV_n(c_s, \underline{c}_b(c_s), s_n(c_s, \underline{c}_b(c_s)))}{dc_s} = \frac{\partial V_n}{\partial c_s} + \frac{\partial V_n}{\partial c_b} \underline{c}_b'(c_s)\bigg|_{(\underline{c}_b, s_n)}.$$

At $c_b = \underline{c}_b$, ω_b and s_n are interior, so $\frac{\partial V_n}{\partial c_s} = -\int_{\omega_b} F(s_n|\omega)dG(\omega) < 0$, and from above, $\frac{\partial V_n}{\partial c_b} > 0$, which gives $\underline{c}_b'(c_s) > 0$. Finally, to show that $\lim_{c_s \to \infty} \underline{c}_b(c_s) = \infty$, We first show that $\lim_{c_s \to \infty} V_n(c_s, c_b, s_n(c_s, c_b))$ exists and is strictly less than $V_r(s_r)$. Note that $\frac{dV_n(c_s, c_b, s_n(c_s, c_b))}{dc_s} = \frac{\partial V_n}{\partial c_s} = -\int_{\omega_b} F(s_n|\omega)dG(\omega) \leq 0$, so the optimized V_n is weakly decreasing in c_s and therefore the limit exists. If $\liminf_{c_s \to \infty} s_n(c_s, c_b) = \underline{s}$, then $\liminf_{c_s \to \infty} \omega_b(s_n, c_b) = \underline{\omega}$, and so $\lim_{c_s \to \infty} V_n(c_s, c_b, s_n(c_s, c_b)) = V_r(\underline{s}) < V_r(s_r)$. If $\liminf_{c_s \to \infty} \omega_b(s_n, c_b) = \bar{\omega}$, then $\lim_{c_s \to \infty} V_n(c_s, c_b, s_n(c_s, c_b)) = 0 <$

$V_r(s_r)$. If $\liminf_{c_s\to\infty} s_n(c_s, c_b) > \underline{s}$ and $\liminf_{c_s\to\infty} \omega_b(s_n, c_b) < \bar{\omega}$, then $\lim_{c_s\to\infty} V_n(c_s, c_b, s_n(c_s, c_b)) = -\infty < V_r(s_r)$. So regardless of the limit behavior of s_n and w_b, $\lim_{c_s\to\infty} V_n(c_s, c_b, s_n(c_s, c_b)) < V_r(s_r)$. Therefore, given $\epsilon > 0$, there exists a $\delta > 0$ such that $V_n(\delta, \epsilon, s_n(\delta, \epsilon)) < V_r(s_r)$, which implies $\underline{c}_b(\delta) > \epsilon$. Because $\underline{c}_b$ is strictly increasing in c_s, for all $c_s > \delta$ we must have $\underline{c}_b(c_s) > \underline{c}_b(\delta) > \epsilon$, and therefore $\lim_{c_s\to\infty} \underline{c}_b(c_s) = \infty$. □

Proof of Proposition 6. Given $c_b \geq 0$, let $\hat{s}(c_b)$ be the highest $s \in S$ such that $\omega_b(s, c_b) = \underline{\omega}$. As c_b increases from zero to infinity, $\hat{s}$ decreases from $\bar{s}$ to $\underline{s}$, so there exists $\hat{c}_b$ such that $\hat{s}(\hat{c}_b) = s_r \in (\underline{s}, \bar{s})$. Denote by $s_n^-(c_s)$ a maximizer of $V_n(c_s, \hat{c}_b, s)$ over $[\underline{s}, s_r]$ and $s_n^+(c_s)$ a maximizer of $V_n(c_s, \hat{c}_b, s)$ over $[s_r, \bar{s}]$. Because $E[v(\omega)|s_r] = 0$, taking the right derivative of V_n with respect to s and evaluating it at $(c_s, c_b, s) = (0, \hat{c}_b, s_r)$ gives

$$\frac{\partial V_n}{\partial s} = -\frac{\partial \omega_b}{\partial s}[1 - F(s_r|\underline{\omega})]v(\underline{\omega})dG(\underline{\omega}) > 0,$$

so there exists $s > s_r$ such that $V_n(0, \hat{c}_b, s) > V_n(0, \hat{c}_b, s_r)$. Therefore, $V_n(0, \hat{c}_b, s_n^+(0)) > V_n(0, \hat{c}_b, s_r) = V_r(s_r)$. By the maximum theorem and the envelope theorem, as c_s increases from zero to infinity, $V_n(\cdot, \hat{c}_b, s_n^+(\cdot))$ decreases continuously from $V_n(0, \hat{c}_b, s_n^+(0)) > V_r(s_r)$ to zero. So there exists $\hat{c}_s > 0$ such that $V_n(\hat{c}_s, \hat{c}_b, s_n^+(\hat{c}_s)) = V_r(s_r)$. Also note that for all $s \in (\underline{s}, s_r]$, $V_n(\hat{c}_s, \hat{c}_b, s) < V_r(s) \leq V_r(s_r)$, and for $s = \underline{s}$, $V_n(\hat{c}_s, \hat{c}_b, \underline{s}) = V_r(\underline{s}) < V_r(s_r)$. Therefore, $V_n(\hat{c}_s, \hat{c}_b, s_n^-(\hat{c}_s)) < V_r(s_r) = V_n(\hat{c}_s, \hat{c}_b, s_n^+(\hat{c}_s))$. Because $V_n(\cdot, \hat{c}_b, s_n^-(\cdot))$ and $V_n(\cdot, \hat{c}_b, s_n^+(\cdot))$ are continuous and strictly decreasing at $\hat{c}_s$, there exists $c_s > \hat{c}_s$ such that $V_n(c_s, \hat{c}_b, s_n^-(c_s)) < V_n(c_s, \hat{c}_b, s_n^+(c_s)) < V_r(s_r)$. Therefore, $s_n(c_s, \hat{c}_b) = s_n^+(c_s) > s_r$ and $V_n(c_s, \hat{c}_b, s_n(c_s, \hat{c}_b)) < V_r(s_r)$. □

Proof of Proposition 7. 1. From the definition of ω_b, $u(\omega_b)\sum_{s\geq s_p} f(s|\omega_r) = c_b \sum_{s<s_p} f(s|\omega_b)$. If $\omega_r = \omega_b$, the result follows immediately. Otherwise, if $\omega_r < \omega_b$, it follows from MLRP and the fact that $u(\omega)$ is increasing in ω that $u(\omega_r)\sum_{s\geq s_p} f(s|\omega_r) < c_b \sum_{s<s_p} f(s|\omega_r)$. The result follows.

2. From part 1, the obedience constraint holds for $\omega = \omega_r$. Moreover,

since $u(\bar{\omega})\sum_{s\geq s_p} f(s|\omega_i|s)h(\bar{\omega}|s) = 0$, the obedience constraint also holds for $\omega_i = \bar{\omega}$. Since the given disclosure rule achieves the highest possible payoff for the regulator $(v(\omega_r, s)\sum_{s\geq s_p} f(s))$, it is optimal.

□

Proof of Proposition 8. As a preliminary note the following:

(i) Instead of solving the original problem, we can solve a relaxed problem in which we replace constraint (14) with the following:

$$\sum_{\omega_i\in\Omega'} h(\omega_i|s) = 1 \text{ for every } s \geq s_p \tag{27}$$

$$\sum_{\omega_i\in\Omega'} h(\omega_i|s) \leq 1 \text{ for every } s < s_p \tag{28}$$

To see why note that if a solution to the relaxed problem is such that $\sum_{\omega_i\in\Omega'} h(\omega_i|s) < 1$ for some $s' < s_p$, we can define an alternate disclosure rule that puts the remaining mass $1 - \sum_{\omega_i\in\Omega'} h(\omega_i|s)$ on the recommendation $\bar{\omega}$ (i.e., not to invest). The alternate rule continues to satisfy all the constraints, and it gives the same value for the objective function.

(ii) Denote $u(\omega_i, s) \equiv u(\omega_i)\infty_{s\geq s_p} - c_b\infty_{s<s_p}$. The Lagrangian of the regulator's problem is

$$\mathcal{L} = \sum_{\omega_i\in\Omega'}\sum_{s\in S}[f(s)v(\omega_i, s) - \lambda_{\omega_i}u(\omega_i, s)f(s|\omega_i)]h(\omega_i|s)ds - \mu_s[\sum_{i=1}^{N} h(\omega_i|s) - 1], \tag{29}$$

where λ_{ω_i} is the Largrange multiplier for the obedience constraint for ω_i and μ_s is the Largrange multiplier of constraint s in (14).

(iii) The dual problem of the (relaxed) problem is as follows:

$$\min_{\{\mu_s\}_{s\in S}, \{\lambda_\omega\}_{\omega\in\Omega'}} \sum_{s\in S} \mu_s \tag{30}$$

subject to

$$\mu_s + u(\omega)f(s|\omega)\lambda_\omega \geq f(s)v(\omega, s) \text{ for every } s \geq s_p \text{ and } \omega \in \Omega' \tag{31}$$

$$\mu_s - c_b f(s|\omega)\lambda_\omega \geq 0 \text{ for every } s < s_p \text{ and } \omega \in \Omega' \tag{32}$$

$$\mu_s \geq 0 \text{ for every } s < s_p$$

$$\lambda_{\underline{\omega}} \leq 0,\ \lambda_{\bar{\omega}} \geq 0.$$

(iv) $\lambda_\omega \geq 0$ for every $\omega > \underline{\omega}$ such that $\sum_{s\in S} h(\omega, s) > 0$. To see why, note that if $\omega > \underline{\omega}$ and $\sum_{s\in S} h(\omega|s) > 0$, there must be $s' < s_p$ such that $h(\omega|s') > 0$. From the FOC for $h(\omega|s')$, $\lambda(\omega)c_b f(s'|\omega) = \mu_{s'} \geq 0$.

We are now ready to prove the proposition.

Part 1. Suppose to the contrary that there exist $s > s_p$ and $\omega_i > \omega_r$ such that $h(\omega_i|s) > 0$. Then from the obedience constraints, there exists $s' < s_p$, such that $h(\omega_i|s') > 0$, and from the FOC for $h(\omega_i|s')$ and $h(\omega_r|s')$,

$$\lambda_{\omega_i} c_b f(s'|\omega_i) = \mu_{s'} \geq \lambda_{\omega_r} c_b f(s'|\omega_r). \tag{33}$$

From (33 and the MLRP property (and since $\lambda \geq 0$), it follows that

$$\lambda_{\omega_i} \geq \frac{f(s'|\omega_r)}{f(s'|\omega_i)}\lambda_{\omega_r} > \frac{f(s|\omega_r)}{f(s|\omega_i)}\lambda_{\omega_r}. \tag{34}$$

Moreover, since $V(\omega_i, s) \leq V(\omega_r, s)$ and $u(\omega_i) > u(\omega_r) \geq 0$, it follows that

$$\begin{aligned} \mu_s &= f(s)V(\omega_i, s) - \lambda_{\omega_i} u(\omega_i) f(s|\omega_i) \\ &< f(s)V(\omega_r, s) - \lambda_{\omega_r} u(\omega_r) f(s|\omega_r). \end{aligned} \tag{35}$$

But this contradicts the FOC for $h(\omega_r|s)$.

Part 2. Consider $\omega_i > \omega_j$ and $s < s' < s_p$, and suppose $h(\omega_i|s) > 0$. From the FOC for $h(\omega_i|s)$ and $h(\omega_j|s)$,

$$\lambda_{\omega_i} c_b f(s|\omega_i) = \mu_s \geq \lambda_{\omega_j} c_b f(s|\omega_j). \tag{36}$$

Hence,

$$\begin{aligned}\lambda_{\omega_i} f(s'|\omega_i) &= \lambda_{\omega_i} f(s'|\omega_i)\frac{f(s|\omega_i)}{f(s|\omega_i)} \geq \lambda_{\omega_j} f(s'|\omega_i)\frac{f(s|\omega_j)}{f(s|\omega_i)} \\ &> \lambda_{\omega_j} f(s'|\omega_i)\frac{f(s'|\omega_j)}{f(s'|\omega_i)} = \lambda_{\omega_j} f(s'|\omega_j).\end{aligned}$$

The first inequality follows from (36) and the second inequality follows from the MLRP property. Finally, from the FOC for $h(\omega_i|s')$, $\lambda_{\omega_i} f(s'|\omega_i) \leq \mu_{s'}$. Hence, $\lambda_{\omega_j} f(s'|\omega_j) < \mu_{s'}$. So from the FOC for $h(\omega_j|s')$, $h(\omega_j|s') = 0$.

Part 3. Consider $\omega_i < \omega_j < \omega_r$ and $s' > s \geq s_p$, and suppose $h(\omega_i|s) > 0$. From the FOC for $h(\omega_i|s)$ and $h(\omega_i|s)$,

$$f(s)V(\omega_i, s) - \lambda_i u(\omega_i) f(s|\omega_i) = \mu_s \geq f(s)V(\omega_j, s) - \lambda_j u(\omega_j) f(s|\omega_j). \tag{37}$$

Rearranging terms, we obtain:

$$\frac{f(s)V(\omega_i, s) - f(s)V(\omega_j, s)}{f(s|\omega_i)} + \lambda_j u(\omega_j)\frac{f(s|\omega_j)}{f(s|\omega_i)} - \lambda_i u(\omega_i) \geq 0 \tag{38}$$

From the FOC for $h(\omega_i|s')$,

$$\mu_s \geq f(s')V(\omega_i, s) - \lambda_i u(\omega_i). \tag{39}$$

Hence, to show that $h(\omega_j|s') = 0$, it is sufficient to show that

$$f(s')V(\omega_i, s) - \lambda_i u(\omega_i) f(s|\omega_i) > f(s')V(\omega_j, s) - \lambda_j u(\omega_j) f(s|\omega_j), \tag{40}$$

or equivalently

$$\frac{f(s')V(\omega_i,s)-f(s')V(\omega_j,s)}{f(s'|\omega_i)}+\lambda_j u(\omega_j)\frac{f(s'|\omega_j)}{f(s'|\omega_i)}-\lambda_i u(\omega_i)>0. \tag{41}$$

Observe that

$$\frac{f(s)V(\omega_i,s)-f(s)V(\omega_j,s)}{f(s|\omega_i)}=\int_{\omega_i}^{\omega_j} v(\omega)\frac{f(\omega|s)f(s)}{f(s|\omega_i)}d\omega=\int_{\omega_i}^{\omega_j} v(\omega)\frac{f(s|\omega)}{f(s|\omega_i)}g(\omega)d\omega \tag{42}$$

From the MLRP property, $\frac{f(s'|\omega_j)}{f(s'|\omega_i)}>\frac{f(s|\omega_j)}{f(s|\omega_i)}$, and for every $\omega>\omega_i, \frac{f(s|\omega)}{f(s|\omega_i)}<\frac{f(s'|\omega)}{f(s'|\omega_i)}$. Hence, (38) implies (41). □

Proof of Proposition 9. From Proposition 8, we can assume, without loss of generality, that the set of recommendations does not include recommendation above ω_r. The dual problem can be written as $\min_{\mu_s,\mu_{s'},\{\lambda_\omega\}_{\omega\in\Omega'}}\mu_s+\mu_{s'}$ subject to

$$\mu_s\geq v(\omega,s)f(s)-u(\omega)f(s|\omega)\lambda_\omega \text{ for every } \omega\in\Omega' \tag{43}$$

$$\mu_{s'}\geq c_b f(s'|\omega)\lambda_\omega \text{ for every } \omega\in\Omega' \tag{44}$$

$$\mu_{s'}\geq 0,\ \ \lambda_{\underline{\omega}}\leq 0.$$

Since $\lambda_{\underline{\omega}}\leq 0$, it follows that $\mu_s\geq v(\underline{\omega},s)f(s)$. We show that there is a solution to the dual problem in which $\lambda_{\underline{\omega}}=0$ and

$$\lambda_\omega=\frac{[v(\omega,s)-v(\underline{\omega},s)]f(s)}{u(\omega)f(s|\omega)},$$

for all other $\omega\neq\underline{\omega}$. Observe that under this solution, $\mu_s=v(\underline{\omega},s)f(s)$ and (43) is binding for every $\omega\in\Omega'$. Moreover, $\mu_{s'}=\max_{\omega\leq\omega_r}c_b f(s'|\omega)\lambda_\omega$. From the definition of $\omega_{s,s'}$, this implies that $\omega_{s,s'}=\arg\max_{\omega\leq\omega_r}c_b f(s'|\omega)\lambda_\omega$, and $\mu_{s'}>c_b f(s'|\omega)\lambda_\omega$, for every $\omega\neq\omega_{s,s'}$.

To show that this is a solution, we show that any other λ_ω weakly increases the sum $\mu_s+\mu_{s'}$. Specifically:

For every ω, setting a higher λ_ω will potentially increase $\mu_{s'}$ and will not decrease μ_s, since we must have $\mu_s \geq v(\underline{\omega}, s)f(s)$. For $\omega = \omega_{s,s'}$, setting a lower λ_ω will increase μ_s and decrease $\mu_{s'}$, but since $\omega_{s,s'} \geq \omega_b$, and since $u(\omega)f(s|\omega) \geq c_b f(s'|\omega)$ for every $\omega \geq \omega_b$, the net effect will be an increase in the sum $\mu_s + \mu_{s'}$. Finally, for every $\omega \neq \omega_{s,s'}$, setting a lower λ_ω will increase μ_s and will have no effect on $\mu_{s'}$. $\square$