Social Insurance, Information Revelation, and Lack of Commitment*

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October 2014

Abstract

We study the optimal provision of insurance against unobservable idiosyncratic shocks in a setting in which a benevolent government cannot commit. A continuum of agents and the government play an infinitely repeated game. Actions of the government are constrained only by the threat of reverting to the worst perfect Bayesian equilibrium (PBE). We construct a recursive problem that characterizes the resource allocation and information revelation on the Pareto frontier of the set of PBE. We prove a version of the Revelation Principle and find an upper bound on the maximum number of messages that are needed to achieve the optimal allocation. Agents play mixed strategies over that message set to limit the amount of information transmitted to the government. The central feature of the optimal contract is that agents who enter the period with low implicitly-promised lifetime utilities reveal no information to the government and receive no insurance against current period shock, while agents with high promised utilities reveal precise information about their current shock and receive insurance as in economies with full commitment by the government.

*We thank Mark Aguiar, Fernando Alvarez, V.V. Chari, Hugo Hopenhayn, Ramon Marimon, Stephen Morris, Chris Sleet, Pierre Yared, Sevin Yeltekin, Ariel Zetlin-Jones for invaluable suggestions and all the participants at the seminars in Bocconi, Carnegie Mellon, EIEF, Georgetown, Minnesota, Norwegian Business School, Paris School of Economics, Princeton, University of Lausanne, University of Vienna, Washington University, the SED 2013, the SITE 2013, the ESSET 2013 meeting in Gerzensee, 12th Hydra Workshop on Dynamic Macroeconomics and 2014 Econometric Society meeting in Minneapolis. Golosov thanks the NSF for support and the EIEF for hospitality. We thank Sergii Kiiashko and Pedro Olea for excellent research assistance.

1 Introduction

The major insight of the normative public finance literature is that there are substantial benefits from using past and present information about individuals to provide them with insurance against shocks and incentives to work. A common assumption of the normative literature is that the government is a benevolent social planner with perfect ability to commit. The more information such a planner has, the more efficiently she can allocate resources.[1]

The political economy literature has long emphasized that such commitment may be difficult to achieve in practice.[2] Self-interested politicians and voters – whom we would broadly refer to as "the government" – are tempted to re-optimize over time and choose new policies. When the government cannot commit, the benefits of providing more information to the government are less clear. Better informed governments may allocate resources more efficiently as in the conventional normative analysis but may also be more tempted to depart from the ex-ante desirable policies. The analysis of such environments is difficult because the main analytical tool to study private information economies – the Revelation Principle – fails when the decision maker cannot commit.

In this paper we study optimal information revelation and resource allocation in a simple model of social insurance – the unobservable taste shock environment of Atkeson and Lucas (1992). This environment, together with closely related models of Green (1987), Thomas and Worrall (1990), Phelan and Townsend (1991), provides theoretical foundation for a lot of recent work in macro and public finance. This set up and its extensions was used to study design of unemployment and disability insurance (Hopenhayn and Nicolini (1997), Golosov and Tsyvinski (2006)), life cycle taxation (Farhi and Werning (2013), Golosov, Troshkin and Tsyvinski (2011)), human capital policies (Stantcheva (2014)), firm dynamics (Clementi and Hopenhayn (2006)), military conflict (Yared (2010)), international borrowing and lending (Dovis (2009)). In the key departure from that literature we assume that resources are allocated by the government, which, although benevolent, lacks commitment. We study how information revelation affects incentives of the government to provide insurance in such settings and characterize the

[1] The seminal work of Mirrlees (1971) started a large literature in public finance on taxation, redistribution and social insurance in presence of private information about individuals' types. Well known work of Akerlof (1978) on "tagging" is another early example of how a benevolent government can use information about individuals to impove efficiency. For the surveys of the recent literature on social insurance and private informaiton see Golosov, Tsyvinski and Werning (2006) and Kocherlakota (2010).

[2] There is a vast literature in political economy that studies frictions that policymakers face. For our purposes, work of Acemoglu (2003) and Besley and Coate (1998) is particularly relevant who argue that inefficiencies in a large class of politico-economic models can be traced back to the lack of commitment. Kydland and Prescott (1977) is the seminal contribution that was the first to analyze policy choices when the policymaker cannot commit.

properties of the optimal contract.

Formally, we study a repeated game between the government and a continuum of atomless citizens in the spirit of Chari and Kehoe (1990) and focus on the perfect Bayesian equilibria (PBE) that are not Pareto-dominated by other PBE. The economy is endowed with a constant amount of perishable good in each period and the government allocates that endowment among agents. Agents receive privately observable taste shocks that follow a Markov process, which is assumed to be iid in the baseline version of our model. Agents transmit information about their shocks to the government by sending messages. The government uses these messages to form posterior beliefs about realization of agents' types and allocate resources. The main friction is that ex-post, upon learning information about the agent's type, the government has the temptation to allocate resources differently from what is required ex-ante to provide incentives to reveal information. The more precise information the government has about the agents' types, the higher its payoff from deviation is.

Our paper makes three contributions. First, we construct a recursive formulation of our problem. The key difficulty that we need to overcome is that the value of the worst equilibrium depends on the implicit promises made to all agents. The standard recursive techniques, that characterize optimal insurance for each history of past shocks in isolation from other histories, do not apply directly since information that any agent reveals affects the government's incentives to renege on the promises made to other agents. We make progress by showing how to construct an upper bound for the value of deviation which coincides with that value if all agents play the best PBE and takes weakly higher value for any other strategy. This upper bound can be represented as a history-by-history integral of functions that depends only on the current reporting strategy of a given agent, and thus can be represented recursively. Since the original best PBE strategies still satisfy all the constraints of this modified, tighter problem, the solution to this recursive problem allows us to uncover the best PBE. The resulting recursive problem is very simple. It is essentially a standard problem familiar from the recursive contract literature with two modifications: agents are allowed to choose mixed rather than pure strategies over their reports and there is an extra term in planner's objective function that captures "temptation" costs of revealing more information.

Our second contribution is to show a version of the Revelation Principle for our settings. It is well known that standard direct revelation mechanisms, in which agents report their types truthfully to the government, are not efficient in this type of settings. Bester and Strausz (2001) showed that, in a repeated game between one principle and one agent whose type can take N values, the message space can be restricted to only N messages over which agents play mixed reporting strategies. These techniques do not apply to economies with more than one agent.[3] We show that in our baseline version of the model with continuum of agents and iid shocks it is without loss of generality to restrict attention to message spaces with at most $2N-1$ messages. This result follows from the fact that with iid shocks our recursive characterization satisfies single-crossing property, and therefore all messages that give any type n the highest utility can be partitioned in three regions: those that also give the highest utility to types $n+1$ and $n-1$ and those that give the highest utility to only type n. There can be at most $2N-1$ of such partitions and convexity of our problem implies that it is sufficient to have one message per each partition.

Our third contribution is the characterization of the properties of the optimal contract and the efficient information revelation. As in the full commitment case of Atkeson and Lucas (1992), it is optimal to allocate resources to agents with temporality high taste shocks by implicitly promising to increase future lifetime utility of agents with temporarily low taste shocks. As agents experience different histories of shocks, there is a distribution of promised utilities at any given time. We show that it is efficient for agents who enter the period with different promised utilities to reveal different amounts of information. In particular, we show that under quite general conditions agents who enter the period with low promised utilities reveal no information about their idiosyncratic shock in that period and receive no insurance. In contrast, under some additional assumption on the utility function and the distribution of shocks, agents who enter the period with high promised utilities reveal their private information fully to the government. The optimal insurance contract for such agents closely resembles the contract when the government can commit.

The intuition for this result can be seen from comparing costs and benefits of revealing information to the government. The costs are driven by the temptation to deviate from the ex-ante optimal plan and re-optimize. When the government deviates from the best equilibrium, it reneges on all the past implicit promises and allocates consumption to each agent based on its posterior beliefs about agent's current type. Therefore, the incentives for the government to deviate depend only on the total amount information it has but not on which agents reveal it. On the contrary, the benefits of information revelation depend on the efficiency gains from better information *on the equilibrium path*, which in turn depend on the implicit promises with which an agent enters the period. The general lesson is that the agents for whom better

[3] See also Skreta (2006) who extends the analysis to an arbitrary type space and show the optimal selling mechanism in a game between one buyer and one seller. Bester and Strausz (2000) provide an example of an economy with two agents to illustrate that that with more than one agent message spaces with only N elements are restrictive.

information leads to the highest efficiency gains on the equilibrium path should send precise signals to the government. In Atkeson and Lucas (1992) multiplicative taste shock environment those are the agents with high promised utility.

Participation constraints of the government prevent the emergence of the extreme inequality, known as immiseration, which is a common feature of environments with commitment. The optimal contract exhibits mean reversion, so that agents with low current promised utilities receive in expectation higher future promises, and vice versa. This implies that in an invariant distribution no agent is stuck in the no-insurance region forever, and in a finite number of steps he reaches a point when he reveals some information about his shock and some insurance is provided. Thus, we show that in the invariant distribution there is generally an endogenous lower bound below which agent's promised utility never falls.

Our paper is related to a relatively small literature on mechanism design without commitment. Roberts (1984) was one of the first to explore the implications of lack of commitment for social insurance. He studied a dynamic economy in which types are private information but do not change over time. More recently, Sleet and Yeltekin (2006), Sleet and Yeltekin (2008), Acemoglu, Golosov and Tsyvinski (2010), Farhi et al. (2012) all studied versions of dynamic economies with idiosyncratic shocks closely related to our economy but made various assumptions on commitment technology and shock processes to ensure that any information becomes obsolete once the government deviates. In contrast, the focus of our paper is on understanding incentives to reveal information and their interaction with the incentives of the government. Our paper is also related to the literature on the ratchet effect, e.g. Freixas, Guesnerie and Tirole (1985), Laffont and Tirole (1988). These authors pointed out that without commitment it is difficult to establish which incentive constraints bind, which significantly inhibited further development of this literature. Some papers have thus focused on the analysis of suboptimal but tractable incentive schemes. We show that the problem substantially simplifies when the principal interacts with a continuum of agents since any agent's report does not affect the aggregate distribution of the signals received by the principal. In this case we can obtain a tight characterization of the optimal information revelation for high and low values of promised utilities.

Our results about efficient information revelation are also related to the insights on optimal monitoring in Aiyagari and Alvarez (1995). In their paper the government has commitment but can also use a costly monitoring technology to verify the agents' reports. They characterize how monitoring probabilities depend on the agents' promised values. Although our environment and theirs are very different in many respects, we both share the same insight that more information should be revealed by those agents for whom efficiency gains from better information are the highest. Work of Bisin and Rampini (2006) pointed out that in general it might be desirable to hide information from a benevolent government in a two period economy. Finally, our recursive formulation builds on the literature on recursive contracts with private information and a patient social planner. Work of Farhi and Werning (2007) provides recursive tools that are particularly useful in our settings.

The rest of the paper is organized as follows. Section 2 describes our baseline environment with iid shocks. Section 3 derives the recursive characterization and shows a version of the Revelation Principle. Section 4 analyzes efficient information revelation and optimal insurance. Section 5 extends our analysis to general Markov shocks.

2 The model

The economy is populated by a continuum of agents of total measure 1 and the government. There is an infinite number of periods, $t = 0, 1, 2, ...$ The economy is endowed with e units of a perishable good in each period. Agent's instantaneous utility from consuming c_t units of the good in period t is given by $\theta_t U(c_t)$ where $U : \mathbb{R}_+ \to \mathbb{R}$ is an increasing, strictly concave, continuously differentiable function. The utility function U satisfies Inada conditions $\lim_{c\to 0} U'(c) = \infty$ and $\lim_{c\to\infty} U'(c) = 0$ and it may be bounded or unbounded. All agents have a common discount factor β. Let $\bar{v} = \lim_{c\to\infty} U(c) / (1-\beta)$ and $\underline{v} = \lim_{c\to 0} U(c) / (1-\beta)$ be the upper and lower bounds on the lifetime expected utility of the agents ($\bar{v}$ and $\underline{v}$ may be finite or infinite). For our purposes it is more convenient to work with utils rather than consumption units. Let $C \equiv U^{-1}$ be the inverse of the utility function.

The taste shock θ_t takes values in a finite set Θ with cardinality $|\Theta|$. For most of the analysis we assume that θ_t are iid across agents and across time, but we relax this assumption in Section 5. Let $\pi(\theta) > 0$ be the probability of realization of $\theta \in \Theta$. We assume that $\theta_1 < ... < \theta_{|\Theta|}$ and normalize $\sum_{\theta\in\Theta} \pi(\theta)\theta = 1$. We use superscript t notation to denote a history of realization of any variable up to period t, e.g. $\theta^t = (\theta_0, ..., \theta_t)$. Let $\pi_t(\theta^t)$ denote the probability of realization of history θ^t. We assume that types are private information.

Each agent is identified with a real number $v \in [\underline{v}, \bar{v}]$ which we soon interpret as a lifetime promised utility. All consumers with the same v are treated symmetrically. The distribution of promises v is denoted by ψ. As it will become clear shortly, the initial distribution ψ gives us a flexible way to trace the Pareto frontier of the set of subgame perfect equilibria.

The government collects reports from agents about their types and allocates consumption subject to the aggregate feasibility in each period. The government is benevolent and its payoff is given by the sum of expected lifetime utilities of all agents.

2.1 The game between the government and agents

The physical structure of our environment corresponds exactly to the model of Atkeson and Lucas (1992). The constrained efficient allocations that they characterize can be achieved by the government which is able to commit in period 0 to infinite period contracts. We focus on the environment without commitment. We model the interaction between the government and individuals along the lines of the literature on sustainable plans (Chari and Kehoe (1990), Chari and Kehoe (1993)). Before formally defining the game we briefly outline its structure. The government does not observes agents' types, so it collects information by having the agents submit messages. More specifically, at the beginning of any period t the government chooses a message set M_t from some space $\mathcal{M}$. All agents observe their types and submits messages $m_t \in M_t$. The government allocates consumption to agents as a function of current and past reports.

To capture the main trade-off in information revelation we assume that the government cannot commit even within a period: it can pick any feasible allocation of resources after collecting agents' reports. This allows for a simple and transparent analysis in our benchmark case of iid shocks. As we discuss in Section 5, much of the analysis carries through to the case when shocks are persistent.[4]

We now formally define our game. We start with information sets. Let S^{t-1} be the summary of information available to the government at the end of period $t-1$. We define this set recursively with $S^{-1} = \psi$. $\mathcal{S}^{t-1}$ is the space of public histories S^{t-1}.

Each period has three stages. In the first stage the government chooses a message set $M_t : \mathcal{S}^{t-1} \to \mathcal{M}$. In the second stage agents send reports about their types. Each agent i observes the realization of his type in period t, $\theta_{i,t}$, and of a payoff irrelevant random variable $z_{i,t}$ uniformly distributed on $Z = [0,1]$. The realizations of $z_{i,t}$ are publicly observable. Each agent chooses a reporting strategy $\tilde{\sigma}_{i,t}$ over M_t as a function of the aggregate history $\left(S^{t-1}, M_t\right)$, his past history of reports and sunspot realizations $h_i^{t-1} = \left(v_i, m_i^{t-1}, z_i^{t-1}\right)$, current realization $z_{i,t}$, and the history of his shocks θ_i^t. Realizations of θ^t are privately observed and we call them *private history*. Realizations of h^{t-1} are publicly observed and we call them *personal history*

[4] A small recent literature on social insurance without commitment sidesteps the possibility that the government may misuse information when it deviates from equilibrium strategies. For example, Acemoglu, Golosov and Tsyvinski (2010) allow agents to stop interacting with the government if it deviates, while Sleet and Yeltekin (2008), Farhi et al. (2012) assume that shocks are iid but there is commitment within the period. All these environment are constructed so that the information about individuals becomes obsolete if the government deviates.

with a space of h^{t-1} denoted by H^{t-1}. To economize on notation, define $\breve{h}^t \equiv \left(v, m^{t-1}, z^t\right)$ and let $\breve{H}^t$ be the space of all $\breve{h}^t$. Formally, the reporting strategy is defined as $\tilde{\sigma}_{i,t} : \mathcal{S}^{t-1} \times \mathcal{M} \times \breve{H}^t \times \Theta^t \to \Delta\left(M_t\right)$ where $\Delta\left(M_t\right)$ is a set of all Borel probability measures on M_t. We assume that all agents i with the same history choose the same strategy $\tilde{\sigma}_{i,t}$, and that the law of large numbers holds so that strategies $\tilde{\sigma}_{i,t}$ generate a distribution of reports σ_t over M_t. To simplify the exposition, we will refer to σ_t as agents' strategy. Finally, in the last stage of the period the government chooses utility allocations u_t for each personal history h^t as a function of $\breve{S}^t = \left(S^{t-1}, M_t, \sigma_t\right)$. The distribution of u_t, σ_t, M_t and S^{t-1} constitute the aggregate history at the beginning of the next period, S^t. Let $\sigma_{G,t}$ be the government strategy in period t and Σ_t and $\Sigma_{G,t}$ be the spaces of feasible σ_t and $\sigma_{G,t}$ respectively. We use boldface letters $\mathbf{x}$ to denote the whole infinite sequence of $\{x_t\}_{t=0}^{\infty}$.

For our purposes it is useful to keep track of two objects, the distribution of the agents' private histories and that of personal histories. For any S^t define $\eta_t(h^t, \theta^t|S^t)$ as the measure of agents with histories $\left(h^t, \theta^t\right)$. It is defined recursively. Let $\eta_{-1} = \psi$. Any Borel set A_t of $[\underline{v}, \bar{v}] \times M^t \times Z^t \times \Theta^t$ can be represented as a product $A_t = A_{t-1} \times B_m \times B_z \times B_\theta$ where A_{t-1} is a Borel set of $H^{t-1} \times \Theta^{t-1}$ and B_m, B_z, B_θ are the m_t-,z- and θ-sections of some Borel set of $M_t \times Z \times \Theta$. The measure $\eta_t\left(h^t, \theta^t|S^t\right)$ over space $H^t \times \Theta^t$ is defined as

$$\eta_t\left(A_t|S^t\right) = \eta_{t-1}\left(A_{t-1}|S^{t-1}\right) \Pr\left(z \in B_z\right) \Pr\left(\theta \in B_\theta\right) \sigma_t\left(B_m|S^{t-1}, M_t, A_{t-1} \times B_z \times B_\theta\right).$$

Let $\mu_t(h^t|S^t)$ be the measure of agents with personal histories h^t, defined for each Borel set A of H^t as

$$\mu_t\left(A|S^t\right) = \int_{\Theta^t} \eta_t\left(A \times \theta^t|S^t\right) d\pi_t\left(\theta^t\right). \tag{1}$$

We require the government's strategy to be feasible, so that it satisfies for any S^t

$$\int_{H^t} C\left(u_t\left(\cdot, S^t\right)\right) d\mu_t\left(\cdot|S^t\right) \leq e. \tag{2}$$

Citizens' strategies induce posterior beliefs p_t over $\Delta\left(\Theta^t\right)$. These beliefs are define for all h^t and S^t and satisfy Bayes' rule, i.e. that for any Borel set A of H^t and any θ^t,

$$\int_A p_t\left(\theta^t|S^t, h\right) \mu_t\left(dh|S^t\right) = \eta_t\left(A \times \theta^t|S^t\right) \text{ whenever } \mu_t\left(A|S^t\right) > 0. \tag{3}$$

Finally, we define continuation payoffs for agents and the government. Any pair $(\boldsymbol{\sigma}, \boldsymbol{\sigma}_G)$ generates a stochastic process for utility allocation $\mathbf{u}$. We use $\mathbb{E}_{(\boldsymbol{\sigma}, \boldsymbol{\sigma}_G)} x_t$ to denote the expectation of the random variable $x_t : S^t \times H^t \times \Theta^t \to \mathbb{R}$ with respect to the probability measures generated by strategies $(\boldsymbol{\sigma}, \boldsymbol{\sigma}_G)$, and $\mathbb{E}_{(\boldsymbol{\sigma}, \boldsymbol{\sigma}_G)}\left[x_t|A\right]$ to denote the conditional expectation given some Borel subset A of $S^t \times H^t \times \Theta^t$. The lifetime utility of agent v is then $\mathbb{E}_{(\boldsymbol{\sigma},\boldsymbol{\sigma}_G)}\left[\sum_{t=0}^{\infty}\beta^t\theta_t u_t|v\right]$. Since the government is benevolent and utilitarian, its payoff is $\mathbb{E}_{(\boldsymbol{\sigma},\boldsymbol{\sigma}_G)}\left[\sum_{t=0}^{\infty}\beta^t\theta_t u_t\right]=\int\mathbb{E}_{(\boldsymbol{\sigma},\boldsymbol{\sigma}_G)}\left[\sum_{t=0}^{\infty}\beta^t\theta_t u_t|v\right]\psi(dv)$.

Definition 1 *A Perfect Bayesian Equilibrium (PBE) is a strategy profile* $(\boldsymbol{\sigma}^*,\boldsymbol{\sigma}_G^*)$ *and a belief system* $\mathbf{p}^*$ *such that conditions (2) and (3) are satisfied, and both agents and the government choose* $\boldsymbol{\sigma}^*$ *and* $\boldsymbol{\sigma}_G^*$ *as their best responses, i.e.*

- *Agents' best response: for every* $S^{T-1},M_T,\breve{h}^T,\theta^T$,

$$\mathbb{E}_{(\boldsymbol{\sigma}^*,\boldsymbol{\sigma}_G^*)}\left[\sum_{t=T}^{\infty}\beta^{t-T}\theta_t u_t\middle|S^{T-1},M_T,\breve{h}^T,\theta^T\right]\geq\mathbb{E}_{(\boldsymbol{\sigma}',\boldsymbol{\sigma}_G^*)}\left[\sum_{t=T}^{\infty}\beta^{t-T}\theta_t u_t\middle|S^{T-1},M_T,\breve{h}^T,\theta^T\right]\text{ for all }\boldsymbol{\sigma}'\in\Sigma,\tag{4}$$

- *Government's best response: for every* $\tilde{S}^T\in\left\{S^{T-1},\breve{S}^T\right\}$,

$$\mathbb{E}_{(\boldsymbol{\sigma}^*,\boldsymbol{\sigma}_G^*)}\left[\sum_{t=T}^{\infty}\beta^{t-T}\theta_t u_t\middle|\tilde{S}^T\right]\geq\int\mathbb{E}_{(\boldsymbol{\sigma}^*,\boldsymbol{\sigma}'_G)}\left[\sum_{t=T}^{\infty}\beta^{t-T}\theta_t u_t\middle|\tilde{S}^T\right]\text{ for all }\boldsymbol{\sigma}'_G\in\Sigma_G.\tag{5}$$

We focus on equilibria that maximize the payoff of the utilitarian government subject to delivering lifetime utility of at least v to an agent that belongs to family v. We call it a best equilibrium and define it as follows

Definition 2 *A triple* $(\boldsymbol{\sigma}^*,\boldsymbol{\sigma}_G^*,\mathbf{p}^*)$ *is a best equilibrium if it is a PBE such that*

$$\mathbb{E}_{(\boldsymbol{\sigma}^*,\boldsymbol{\sigma}_G^*)}\left[\sum_{t=0}^{\infty}\beta^t\theta_t u_t\middle|v\right]\geq v\text{ for all }v,\tag{6}$$

and there is no other PBE $(\boldsymbol{\sigma}',\boldsymbol{\sigma}'_G,\mathbf{p}')$ *that satisfies (6) and delivers a higher payoff to the government,*

$$\mathbb{E}_{(\boldsymbol{\sigma}',\boldsymbol{\sigma}'_G)}\left[\sum_{t=0}^{\infty}\beta^t\theta_t u_t\right]>\mathbb{E}_{(\boldsymbol{\sigma}^*,\boldsymbol{\sigma}_G^*)}\left[\sum_{t=0}^{\infty}\beta^t\theta_t u_t\right].$$

Throughout the paper we assume that ψ is such that a best equilibrium exists. We further assume that ψ is such that (6) holds with equality in a best equilibrium. This assumption simplifies the recursive formulation since all the "promise keeping" constraints can be defined as equalities in that case. Without this assumption our period 0 recursive formulation would have to be defined as an inequality, which would make notation more bulky but not affect any of our results.

3 The recursive problem and the Revelation principle

In this section we discuss two intermediate results that are necessary for our analysis of optimal information revelation. First, we derive the recursive problem solution to which characterizes the optimal reporting strategy and consumption allocation in best PBEs. Recursive techniques are the standard tool to solve dynamic contracting problems, which essentially allow to solve for optimal insurance after any given history of shocks separately from the other histories. In our game-theoretic settings the standard approach to constructing Bellman equations is not applicable: the government's incentive to deviate from a best equilibrium depends on the reporting strategies of all agents making the usual history-by-history separation of incentives is impossible. To overcome this difficulty, we consider a constrained problem in which we replace the function that characterizes the value of deviation for the government with another function which is linear in the distribution of the past histories of reports. We show that this function can be chosen such that a solution to the original problem is also a solution to the constrained problem. The advantage is that the linearity of the constrained problem in the distribution of past reports allows us to solve for the optimal contract recursively, history-by-history. The key insight for this construction is that the Lagrange multiplier on resource constraint in the best one shot deviation from best equilibrium strategies summarizes all relevant information.

The second result of this section characterizes the minimum dimension of the message space $\mathbf{M}$ that is needed to support best PBEs. Reduction of an arbitrarily message space $\mathbf{M}$ to a smaller dimension object is important for tractability of the maximization problem. This step is similar in spirit to invoking the Revelation Principle in standard mechanism design problems with commitment, and we refer to this result as the (generalized) Revelation principle.

Although these steps are crucial for our ultimate goal – characterization of efficient information revelation – these tools are of interest in themselves. A lot of environments without commitment share the feature that value of deviation for the "principal" depends on actions of many agents, and the recursive techniques developed in our settings should also be applicable to such environments. To the best of our knowledge, there is no version of the Revelation Principle for economies with multiple agents when the principal lacks commitment. A well known paper by Bester and Strausz (2001) developed a version the Revelation Principle for an economy with one agent, but their approach and the main result do not extend to multi-agent settings.

3.1 The recursive problem

We start by describing a PBE that delivers the lowest payoff to the government, to which we refer as the worst PBE. It is easy to show that there are worst equilibria in which no information is revealed.[5] To see this, let $m^{\varnothing}(M)$ be an arbitrary message from a set $M \in \mathcal{M}$. Consider a strategy profile $(\boldsymbol{\sigma}^w, \boldsymbol{\sigma}_G^w)$ in which all agents report message $m^{\varnothing}(M)$ with probability 1 for all M, and the government allocates the same utility $u_t^w = C^{-1}(e)$ for any message m it receives. The beliefs $\mathbf{p}^w$ that satisfy (3) are given by $p_t^w\left(\theta^t | h^t, S^t\right) = \pi_t\left(\theta^t\right)$ for all $\left(h^t, S^t\right)$.

Lemma 1 $(\boldsymbol{\sigma}^w, \boldsymbol{\sigma}_G^w, \mathbf{p}^w)$ *is a worst PBE.*

Proof. The best response of the government to agents' reporting strategy $\boldsymbol{\sigma}^w$ is to allocate the same utility to all agents since its beliefs $\mathbf{p}^w$ do not depend on m and C is strictly convex. Given this allocation rule, playing $\boldsymbol{\sigma}^w$ is optimal for agents, therefore $(\boldsymbol{\sigma}^w, \boldsymbol{\sigma}_G^w, \mathbf{p}^w)$ is a PBE that delivers the government payoff $U(e)/(1-\beta)$. Strategy $\boldsymbol{\sigma}_G^w$ is feasible for any reporting strategy that agents choose, so the government can attain payoff $U(e)/(1-\beta)$ in any PBE, which proves that $(\boldsymbol{\sigma}^w, \boldsymbol{\sigma}_G^w, \mathbf{p}^w)$ is a worst PBE for the government. ■

Standard arguments establish that best equilibria are supported to reverting to the worst PBE following any deviation of the government from its equilibrium strategies. We split our problem in two parts. First, we fix any sequence of message sets $\mathbf{M}$ and describe the maximization problem that characterizes the best PBE given that sequence of message sets, which we call the *best equilibria given* $\mathbf{M}$. We then show that the same payoff can be achieved with a much simpler message set that has only a finite number of messages. That set is independent of $\mathbf{M}$ and hence without loss of generality it can be chosen as the message set in the best PBE.

We start with several preliminary observations that simplify the notation. Since there is no aggregate uncertainty the aggregate distribution of reports and allocations is a deterministic sequence. Therefore, it is redundant to keep track of the aggregate history S^t in best equilibria and we drop this explicit dependence from our notation. Moreover, without loss of generality we can restrict attention to reporting strategies σ_t that depend on h^{t-1}, z_t and θ_t, but not on the past history of shocks θ^{t-1}.[6] This results holds more generally with Markov shocks, since as long as θ_t follows a Markov process, θ^{t-1} is payoff irrelevant and agents with the same

[5] In fact, a stronger statement is true that in any worst equilibrium no information is revealed. This statement follows from our Lemma 2.

[6] In the Supplementary Material we show that for any PBE $(\boldsymbol{\sigma}', \boldsymbol{\sigma}'_G, \mathbf{p}')$ there is another PBE $(\boldsymbol{\sigma}, \boldsymbol{\sigma}_G, \mathbf{p})$ which delivers the same payoff to all agents and the government as $(\boldsymbol{\sigma}', \boldsymbol{\sigma}'_G, \mathbf{p}')$ (i.e. $(\boldsymbol{\sigma}, \boldsymbol{\sigma}_G, \mathbf{p})$ is *payoff-equivalent* to $(\boldsymbol{\sigma}', \boldsymbol{\sigma}'_G, \mathbf{p}')$) and has the property that σ_t does not depend on θ^{t-1}.

$\left(\breve{h}^t, \theta_t\right)$ can replicate each other strategies for all past θ^{t-1}. This implies that all information about agents at the beginning of period t can be summarized by the distribution of reports μ_{t-1}.

We now turn to setting up the maximization problem that characterizes best equilibria given $\mathbf{M}$. Since any deviation by the government from a best equilibrium triggers a switch to the worst PBE, it is sufficient to consider only the best one-shot deviation from equilibrium strategies by the government. The government that observes the distribution of personal histories μ_t can achieve the maximum payoff $\tilde{W}_t\left(\mu_t\right)$ defined as

$$\tilde{W}_t\left(\mu_t\right) = \sup_{\{u(h)\}_{h \in H^t}} \int_{H^t} \mathbb{E}_{\boldsymbol{\sigma}^t}\left[\theta | h\right] u\left(h\right) d\mu_t \tag{7}$$

subject to

$$\int_{H^t} C\left(u\left(h\right)\right) d\mu_t \leq e. \tag{8}$$

Therefore, any best equilibrium given $\mathbf{M}$ is a solution to

$$\max_{\mathbf{u}, \boldsymbol{\sigma}} \mathbb{E}_{\boldsymbol{\sigma}} \sum_{t=0}^{\infty} \beta^t \theta_t u_t \tag{9}$$

subject to (2),

$$\mathbb{E}_{\boldsymbol{\sigma}} \sum_{s=t}^{\infty} \beta^{s-t} \theta_s u_s \geq \tilde{W}_t\left(\mu_t\right) + \frac{\beta}{1-\beta} U\left(e\right) \text{ for all } t, \tag{10}$$

$$\mathbb{E}_{\boldsymbol{\sigma}} \sum_{t=0}^{\infty} \beta^t \theta_t u_t \geq \mathbb{E}_{\boldsymbol{\sigma}'} \sum_{t=0}^{\infty} \beta^t \theta_t u_t \text{ for all } \boldsymbol{\sigma}' \tag{11}$$

and

$$\mathbb{E}_{\boldsymbol{\sigma}}\left[\left.\sum_{t=0}^{\infty} \beta^t \theta_t u_t \right| v\right] = v. \tag{12}$$

The main difficulty in the analysis of problem (9) is that the sustainability constraint (10) depends on $\tilde{W}_t$, which is a non-linear function of the past reports embedded in the probability distribution μ_{t-1}. To make progress, we replaced $\tilde{W}_t$ with a function that (i) is linear in μ_{t-1}, (ii) is weakly greater than $\tilde{W}_t$ for all μ_t, and (iii) equals $\tilde{W}_t$ at a best equilibrium's distribution of reports. This defines a more constrained maximization problem for which the best equilibrium is still a solution. The linearity of the constraining function in μ_{t-1} allows us to write the constrained problem recursively.

Let $(\mathbf{u}^*, \boldsymbol{\sigma}^*)$ be a solution to (9) and $\boldsymbol{\mu}^*$ be the distribution of personal histories induced by $\boldsymbol{\sigma}^*$. Let $\lambda_t^w \geq 0$ be a Lagrange multiplier in problem (7) given μ_t^*. For any mapping $\sigma : \Theta \rightarrow$

$\Delta(M_t)$ define

$$W_t(\sigma) = \max_{\{u(m)\}_{m\in M_t}} \int_{M_t\times\Theta} (\theta u(m) - \lambda_t^w C(u(m)))\,\sigma(dm|\theta)\,\pi(d\theta) + \lambda_t^w e. \tag{13}$$

We use $\{u^w(m)\}_{m\in M_t}$ to denote the solution to (13), which is given by equation $\lambda_t^w C'(u^w(m)) = \mathbb{E}_\sigma[\theta|m]$.

Function W_t plays an important role in our analysis: We show that $\int W_t(\sigma_t)\,dz_t d\mu_{t-1}$ is an upper bound for $W_t(\mu_t)$ that satisfies the three properties needed for recursive characterization. Before formally proving this result, it is useful to introduce some definitions. We say that reporting strategies σ are *uninformative* if $\mathbb{E}_\sigma[\theta|m] = \sum_\theta \pi(\theta)\theta = 1$ a.s., i.e. for almost every message m with respect to the measure generated by σ. Uninformative reporting strategies reveal no additional information other than the unconditional expectation of θ. All other strategies are called *informative*. We also need to generalize the notion of the envelope theorem for W_t. For any σ and σ' define a derivative of $W_t(\sigma)$ in the direction σ' as

$$\frac{\partial W_t(\sigma)}{\partial\sigma'} \equiv \lim_{\alpha\downarrow 0} \frac{W_t((1-\alpha)\sigma + \alpha\sigma') - W_t(\sigma)}{\alpha}.$$

With these definitions we can state the following lemma, which is important for all subsequent analysis.

Lemma 2 *The multiplier λ_t^w is uniformly bounded away from 0 and belongs to a compact set and, therefore, $u^w(m)$ belongs to a compact set in the interior of $[(1-\beta)\underline{v}, (1-\beta)\bar{v}]$ for all σ. Family of $\{W_t\}_t$ and all families of its directional derivaties are uniformly bounded.*

Function W_t is well defined, continuous, convex, and is minimized if and only if σ is uninformative. For any (σ_t, μ_{t-1})

$$\tilde{W}_t(\mu_t) \le \int_{\breve{H}^t} W_t\left(\sigma_t\left(\cdot|\breve{h}^t,\cdot\right)\right) dz d\mu_{t-1} \tag{14}$$

with equality if $(\sigma_t, \mu_{t-1}) = (\sigma_t^*, \mu_{t-1}^*)$.

The derivative $\partial W_t(\sigma)/\partial\sigma'$ is well-defined for all σ' and satisfies

$$\frac{\partial W_t(\sigma)}{\partial\sigma'} = \int_{M_t\times\Theta} (\theta u^w(m) - \lambda_t^w C(u^w(m)))\left[\sigma'(m|\theta) - \sigma(m|\theta)\right] dm\pi(d\theta). \tag{15}$$

Consider now a modified maximization problem (9) in which we replace (10) with

$$\mathbb{E}_{\boldsymbol{\sigma}} \sum_{s=t}^{\infty} \beta^{s-t}\theta_s u_s \ge \int_{\breve{H}^t} W_t\left(\sigma_t\left(\cdot|\breve{h}^{t-1},\cdot\right)\right) dz d\mu_{t-1} + \frac{\beta}{1-\beta}U(e). \tag{16}$$

The constraint set is tighter in the modified problem, but $(\mathbf{u}^*, \boldsymbol{\sigma}^*)$ still satisfies all the constraints, therefore $(\mathbf{u}^*, \boldsymbol{\sigma}^*)$ is a solution to the modified problem. One can then use standard techniques (see, e.g. Marcet and Marimon (2009) and Sleet and Yeltekin (2008)) and use Lagrangian multipliers to re-write it as

$$\mathcal{L} = \max_{\mathbf{u},\boldsymbol{\sigma}} \mathbb{E}_{\boldsymbol{\sigma}} \sum_{t=0}^{\infty} \bar{\beta}_t \left[\theta_t u_t - \zeta_t C\left(u_t\right) - \chi_t W_t\right] \tag{17}$$

subject to (11) and (12) for some non-negative sequences $\left\{\bar{\beta}_t, \chi_t, \zeta_t\right\}_{t=0}^{\infty}$, with a property that $\hat{\beta}_t \equiv \bar{\beta}_t/\bar{\beta}_{t-1} \geq \beta$, with strictly inequality if and only if constraint (16) binds in period t.[7] We interpret $\hat{\beta}_t$ as the effective discount factor of the government in the best equilibrium. It is greater than agents' discount factor β because the government also needs to take into account how its actions affects the sustainability constraints in the future. This is a general result about the role of the lack of commitment on discounting that was highlighted, for example, by Sleet and Yeltekin (2008) and Farhi et al. (2012).

The maximization problem (17) shows costs and benefits of information revelation. The more information the agents reveal, the easier it is for the planner to maximize the cost-weighted utility function of the agents, $\theta_t u_t - \zeta_t C\left(u_t\right)$. At the same time, better information also increases the temptation to deviate, captured by the term $-\chi_t W_t$. This trade-off is at the heart of our characterization in Section 4.

We make two assumptions on the properties of these Lagrange multipliers which we maintain throughout the analysis, $\limsup \chi_t > 0$ and $\liminf \zeta_t > 0$. These conditions simplify technical arguments and in the Supplementary material we provide sufficient conditions for C that ensure that these assumptions are satisfied. Since the Langrange multipliers should satisfy these properties much more generally – for example whenever the economy converges to an invariant distribution and the long-run immiseration is a feature of the optimal contract with full commitment – we opted to make assumptions on the multipliers directly.[8]

An important simplifying feature of the modified maximization problem and the Lagrangian (17) is that the distribution of past reports μ_{t-1} enters the objective function and the constraints linearly. This allows us to solve for the optimal allocations and reporting strategies separately for each history h^{t-1}. We do so by extending the recursive techniques developed by Farhi and Werning (2007) who study an economy with commitment but in which the principal is more patient than agents.

[7] This representation is possible since allowing the government and agents to condition their strategies on z convexifies the maximization problem (9). The formal arguments are straightforward but cumbersome, we present their sketch in the Supplementary Material.

[8] Immiseration is a common feature of long-term contracts with commitment. See Ljunqvist and Sargent (2004) for a textbook discussion and Phelan (2006) and Hosseini, Jones and Shourideh (2013) for recent contributions to the literature. We discuss existence of the invariant distribution in Section 5.1.

Let $k_0(v)$ be the value of the objective function (17) for family v, defined as

$$k_0(v) = \frac{1}{\bar{\beta}_0} \max_{\mathbf{u},\boldsymbol{\sigma}} \mathbb{E}_{\boldsymbol{\sigma}} \left[\sum_{t=0}^{\infty} \bar{\beta}_t \left(\theta_t u_t - \zeta_t C(u_t) - \chi_t W_t \right) \Bigg| v \right] \tag{18}$$

subject to (11) and (12). The Lagrangian defined in (17) satisfies $\mathcal{L} = \int \bar{\beta}_0 k_0(v) \psi(dv)$. Similarly we can define[9]

$$k_t(v) = \frac{1}{\bar{\beta}_t} \max_{\mathbf{u},\boldsymbol{\sigma}} \mathbb{E}_{\boldsymbol{\sigma}} \left[\sum_{s=0}^{\infty} \bar{\beta}_{t+s} \left(\theta_s u_s - \zeta_{t+s} C(u_s) - \chi_{t+s} W_s \right) \Bigg| v \right] \tag{19}$$

subject to (11) and (12). The next proposition shows the relationship between $k_t(v)$ and $k_{t+1}(v)$.

Proposition 1 *The value function $k_t(v)$ is continuous, concave, differentiable with $\lim_{v \to \bar{v}} k_t'(v) = -\infty$. If utility is unbounded below, then $\lim_{v \to -\infty} k_t'(v) = 1$; if utility is bounded below, then $\lim_{v \to \underline{v}} k_t'(v) = \infty$. The value function $k_t(v)$ satisfies the Bellman equation*

$$k_t(v) = \max_{\substack{\{u(m,z), w(m,z), \sigma(\cdot|z,\theta)\}_{(m,z) \in M_t \times Z} \\ \sigma(\cdot|z,\theta) \in \Delta(M_t)}} \mathbb{E}_\sigma \left[\theta u - \zeta_t C(u) + \hat{\beta}_{t+1} k_{t+1}(w) - \chi_t W_t \right] \tag{20}$$

subject to

$$v = \mathbb{E}_\sigma \left[\theta u + \beta w \right] \tag{21}$$

$$\mathbb{E}_\sigma \left[\hat{\theta} u + \beta w | \hat{\theta}, z \right] \geq \mathbb{E}_{\sigma'} \left[\hat{\theta} u + \beta w | \hat{\theta}, z \right] \text{ for all } z, \hat{\theta}, \sigma' \tag{22}$$

Without loss of generality, the optimum is achieved by randomization between at most two points, i.e. one can choose $\bar{z} \in [0,1]$ and set $(u(\cdot, z'), w(\cdot, z'), \sigma(\cdot|z', \cdot)) = (u(\cdot, z''), w(\cdot, z''), \sigma(\cdot|z'', \cdot))$ for all $z', z'' \leq \bar{z}$ and all $z', z'' > \bar{z}$.

Most of the proof of this proposition follows the arguments of Farhi and Werning (2007) and is provided in the Supplementary materials. The solution to this Bellman equation is attained by some function $g_t^u(v, \theta, z)$, $g_t^w(v, \theta, z)$ and $g_t^\sigma(v, \theta, z)$. We can use these policy functions to generate recursively u_t and σ_t. Standard arguments establish when these $(\mathbf{u}, \boldsymbol{\sigma})$ are a solution to the original problem (9).

Proposition 2 *If $(\mathbf{u}, \boldsymbol{\sigma})$ obtains a maximum to (18), then it is generated by $(\mathbf{g}^u, \mathbf{g}^w, \mathbf{g}^\sigma)$. If $(\mathbf{u}, \boldsymbol{\sigma})$ generated by $(\mathbf{g}^u, \mathbf{g}^w, \mathbf{g}^\sigma)$ is such that*

$$\lim_{t \to \infty} \mathbb{E}_{\boldsymbol{\sigma}} \beta^t v_t = 0 \tag{23}$$

then it is a solution to (18).

Condition (23) can be verified in the same way as in Farhi and Werning (2007).

[9] Variables $(\mathbf{u}, \boldsymbol{\sigma}) = \{u_t, \sigma_t\}_{t=0}^\infty$ are defined over slightly difference spaces in problems (18) and (19). The sequence of message sets is $\{M_t\}_{t=0}^\infty$ in (18), while it is $\{M_{s+t}\}_{t=0}^\infty$ in (19).

3.2 The Revelation Principle

The recursive problem (20) is similar to the usual recursive dynamic contract formulation with commitment with two modifications. First, agents play a mixed reporting strategy over set M rather than a pure reporting strategy over set Θ. Second, there is an additional term $-\chi_t W_t$ that captures the cost of information revelation. In this section we further simplify the problem by showing that incentive constraints (22) impose a lot of structure on agents' reporting strategies and simplify the message space that needs to be considered in best PBEs.

Since incentive constraints (22) hold for each z, we drop in this section the explicit dependence of strategies on the payoff irrelevant variable, and our arguments should be understood to apply for any z. Let $\{u(m), w(m)\}_{m \in M_t}$ be any allocation that satisfies (22). Let $M_t(\theta) \subset M_t$ be the subset of messages that give type θ the highest utility, i.e.

$$\theta u(m) + \beta w(m) \geq \theta u(m') + \beta w(m') \text{ for all } m \in M_t(\theta), m' \in M_t.$$

Sum this inequality with an analogous condition for type θ' to get

$$(\theta - \theta') u(m) \geq (\theta - \theta') u(m') \text{ for all } \theta, \theta', m \in M_t(\theta), m' \in M_t(\theta'). \tag{24}$$

One immediate implication of this equation is that any incentive compatible allocation is monotone in types, i.e. $\theta > \theta'$ implies $u(m) \geq u(m')$ and $w(m) \leq w(m')$. It follows that each message set $M_t(\theta_i)$ can be partitioned into three regions: the messages that also belong to $M_t(\theta_{i+1})$, the messages that also belong to $M_t(\theta_{i-1})$, and the messages that do not belong to any other $M_t(\theta_j)$.

The next proposition shows that it is without loss of generality to assume that agents send only one message per each partition. Since there can be at most $2|\Theta| - 1$ of such partitions, it also gives an upper bound on the cardinality of the message space needed in best PBEs.

Proposition 3 *Any best PBE is payoff equivalent to a PBE in which agents report no more than $2|\Theta| - 1$ messages after any history $\breve{h}^t$. Furthermore such PBE can be constructed so that if $m' \neq m''$ then the utility allocation in equilibrium, u_t, satisfies $u_t(\breve{h}^t, m) \neq u_t(\breve{h}^t, \hat{m})$.*

We briefly sketch the key steps of the arguments. Consider any partition $M_t(\theta_i) \cap M_t(\theta_{i+1})$. Equation (24) implies that $(u(m), w(m))$ must be the same for all $m \in M_t(\theta_i) \cap M_t(\theta_{i+1})$. Therefore, all reporting strategies σ that keep the same probability of reporting messages $m \notin M_t(\theta_i) \cap M_t(\theta_{i+1})$ have the same on-the-equilibrium-path payoff $\mathbb{E}_\sigma\left[\theta u - \zeta_t C(u) + \hat{\beta}_{t+1} k_{t+1}(w)\right]$. Thus, in a best BPE the reporting of messages in $M_t(\theta_i) \cap M_t(\theta_{i+1})$ must minimize the off-the-equilibrium-path payoff $\mathbb{E}_\sigma[\chi_t W_t]$. Convexity of W_t implies that such reporting strategy keeps posterior beliefs of the government the same for all $m \in M_t(\theta_i) \cap M_t(\theta_{i+1})$. Then it is without loss of generality to allow only one message for each $M_t(\theta_i) \cap M_t(\theta_{i+1})$.

The reverse arguments allow to reach the same conclusion for the subset of $M_t(\theta_i)$ that does not intersect with any other $M_t(\theta_j)$, which we denote $M_t^{excl}(\theta_i)$. Conditional of observing any message sent with a positive probability from subset $M_t^{excl}(\theta_i)$, the government learns that the sender is type θ_i with certainty. Therefore the off-the-equilibrium-path payoff does not depend on the exact probability with each type θ_i reports messages in $M_t^{excl}(\theta_i)$. Strict convexity of the on-the-equilibrium-path payoff implies that the government chooses that same $(u(m), w(m))$ for all $m \in M_t^{excl}(\theta_i)$, which again implies that it is without loss of generality to restrict attention to only one message.

Proposition 3 is very useful for the characterization of the optimal incentive provision and information revelation in the next section. It also allows us to prove a version of the Revelation Principle in our economy. Let M_Θ be any set of cardinality $|\Theta|$.

Corollary 1 *Any best PBE given* $\mathbf{M}$ *is payoff equivalent to a best PBE given* $M_t = M_\Theta$ *for all* t. *The incentive constraint (22) for such equilibrium can be written, for each* z, *as* $\sigma(m_{2i-1}|\theta_{2i-1}) > 0$ *and*

$$\theta_i u(m_{2i-1}) + \beta w(m_{2i-1}) \geq \theta_i u(m) + \beta w(m), \tag{25}$$

$$\sigma(m|\theta_i)\left[(\theta_i u(m_{2i-1}) + \beta w(m_{2i-1})) - (\theta_i u(m) + \beta w(m))\right] = 0 \text{ for all } i = 1, .., |\Theta|, m \in M_\Theta.$$

This result shows that in general the dimensionality of message space with a continuum of agents is finite but bigger than the dimensionality of the message space with only one agent. In the latter case Bester and Strausz (2001) showed that without loss of generality one can restrict attention to a message space $M = \Theta$. Their result does not extend to economies with multiple agents.[10] Corollary 1 shows that even with a continuum of agents the dimensionality of the message space remains small.

4 Characterization

In this section we characterize properties of the efficient information revelation. We start with a simplified environment that illustrates the main trade-offs very transparently. We then extend the insights developed by this example to our general settings.

[10] See Bester and Strausz (2000) for a counterexample with two agents.

4.1 A simplified environment

We make two simplifying assumptions in this section. First, we assume that the government can deviate from its implicit promises in period t only if it deviates in period 0. This simplifies the analysis of the best equilibrium as it is sufficient to ensure that government's sustainability constraint (10) holds only in period 0. This case is isomorphic to our Lagrangian formulation with $\chi_t = 0$ for all $t > 0$. Secondly, we assume that agents' preferences can be represented by a utility function $U(c) = ac^{1/a}$ when $1/a < 1$, with $U(c) = \ln c$ when $1/a = 0$.

Under the first simplifying assumption the Lagrangian (17) can be written, up to a constant, as

$$\mathcal{L} = \max_{\mathbf{u},\boldsymbol{\sigma}} \mathbb{E}_{\boldsymbol{\sigma}} \left\{ -\sum_{t=0}^{\infty} \beta^t \zeta_t C(u_t) - \chi_0 W_0 \right\}$$

subject to (11) and (12). First, consider the sub-game starting from $t = 1$. Since the sustainability constraints do not bind for all $t \geq 1$, the standard Revelation principle applies and the maximization problem that characterizes the optimal allocations is identical to that in Atkeson and Lucas (1992). It is easy to show that the value function with CRRA preferences takes the form $\kappa_1(v) = \mathit{const} \cdot |v|^a$ when $1/a \neq 0$ and $\kappa_1(v) = \mathit{const} \cdot \exp(v)$ when $1/a = 0$.

For any reporting strategy $\sigma \in \Delta(M_0)$ in period 0, let $\kappa_0(v;\sigma)$ be defined as

$$\kappa_0(v;\sigma) = \max_{\{u(m),w(m)\}_{m\in M_0}} \mathbb{E}_\sigma \left[-\zeta_0 C(u) + \beta\kappa_1(v)\right]$$

subject to (21) and (22). Function $\kappa_0(v;\sigma)$ is interpreted as the value function for the government *on the equilibrium path* if an agent plays reporting strategy σ. It inherits the homogeneity property of function κ_1, namely $\kappa_0(v;\sigma) = d(\sigma)|v|^a$ for some constant $d(\sigma)$.

If no public randomization is allowed, the actual value function $\kappa_0(v)$ is given by

$$\kappa_0(v) = \max_\sigma \kappa_0(v;\sigma) - \chi_0 W_0(\sigma).$$

With public randomization the maximum is taken over the convex hull of the function on the right hand side.

Consider first the case when $|\Theta| = 2$ and agents can only play two reporting strategies: a full information revelation strategy σ^{in}, when each type reports a distinct message with probability 1, and a no information revelation strategy σ^{un}, when each type randomizes between all messages with the same probability. Since the former strategy has higher payoff both on- and off- the equilibrium path, $d\left(\sigma^{in}\right) > d(\sigma^{un})$ and $W_0\left(\sigma^{in}\right) > W_0(\sigma^{un})$.

Figure 1 plots $\kappa_0(v;\sigma) - \chi_0 W_0(\sigma)$ for $\sigma \in \left\{\sigma^{in}, \sigma^{un}\right\}$, as well as $\kappa_0(v)$ with public randomization. The key observation is that function $\kappa_0\left(\cdot;\sigma^{in}\right) - \chi_0 W_0\left(\sigma^{in}\right)$ intersects $\kappa_0(\cdot;\sigma^{un}) -$

$\chi_0 W_0(\sigma^{un})$ only once, from below at some point v_2. This shows that the uninformative strategy σ^{un} gives higher value to the government than σ^{in} for all $v < v_2$ and lower value for all $v > v_2$. Allowing public randomization may further increase the government's payoff and create an intermediate region (region (v_1, v_3) on the graph) in which the reporing strategy is stochastic but it does not alter the main conclusion: no information revelation is optimal for low values of v and full information is optimal for high values of v.

To understand the intution for this result, it is useful to compare the on the equilibium path benefits of information revelation, $\kappa_0(\cdot;\sigma^{in}) - \kappa_0(\cdot;\sigma^{un})$, to the off the equilibrium path costs $W_0(\sigma^{in}) - W_0(\sigma^{un})$. Given our CRRA assumption, it is easy to see that the benefits are monotonically increasing in v, equal to zero as $v \to \underline{v}$ and to infinity as $v \to \bar{v}$. The off the equilibrium path costs of information revelation do not depend on v. Therefore, there must exist a region of low enough v where the costs of better information revelation exceed the benefits and a region of high enough v where the opposite conclusion holds.

This conclusion does not depend on the fact that we restricted attention to only two reporting strategies. When $|\Theta| = 2$, any σ that reveals some but not all information to the government satisfies $d(\sigma^{in}) > d(\sigma)$ and $W_0(\sigma) > W_0(\sigma^{un})$. This implies that reporting strategies σ^{un} and σ^{in} have higher payoff than σ for low and high values of v respectively.

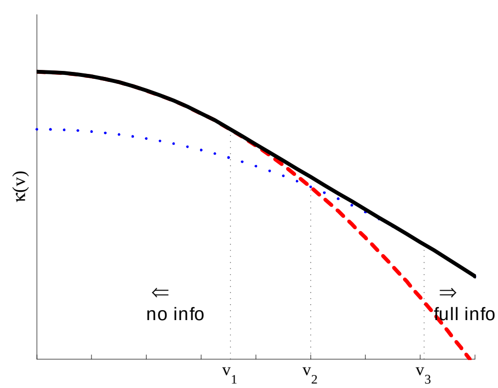

Figure 1: Value functions in the simplified example. Solid line: $\kappa_0(v)$ with public randomization, dotted: $\kappa_0^{in}(v) - \chi_0 W(\sigma_0^{in})$, dashed: $\kappa_0^{un}(v) - \chi_0 W(\sigma_0^{un})$.

When $|\Theta| > 2$, it is still true that $W_0(\sigma) > W_0(\sigma^{un})$ and therefore no information revelation is optimal for low enough v. However, full information revelation no longer gives *strictly* higher value than any other reporting strategy on the equilibrium path. The reason for it is that bunching might be optimal even in standard mechanism design problems with commitment. Our result for high values of v extends if the distribution of types is such that no bunching is desirable with commitment (equation (33) provides a sufficient condition for that). More generally, it can be shown that some information revelation is desirable for high v, in particular it is optimal for θ_1 to reveal his type perfectly.

Finally, we comment on the role of the assumption of constant relative risk aversion. Our arguments for no information revelation for low values of v only used the fact that σ^{un} provides the lowest value to the government off the equilibrium path. This is true for any preferences. The arguments for efficiency of full information revelation rely on the fact that $\kappa_0\left(v;\sigma^{in}\right) - \kappa_0\left(v;\sigma\right)$ becomes unboundedly large for high values of v. A sufficient condition for this is that the absolute risk aversion goes to zero as consumption goes to infinity, and our conclusion holds for all preferences satisfying this condition.

This example illustrates the general principle that goes beyond the particular taste-shock model that we consider in this paper. Since the off the equilibrium path costs of information revelation do not depend on past promises (the principal simply reneges on them when deviates) but the benefits do, it is optimal to reveal less information for those agents for whom the on the equilibrium gains are lowest. The gains from better information are monotone in v in the taste shock economy that we consider in this paper, which allows us to get sharp cut-offs, but the general principle holds regardless of this property.

4.2 Characterization of the general case

In this section we study efficient information revelation in our baseline model of Section 3. Let (u_v, w_v, σ_v) be a solution to (20). To streamline the exposition, in the body of the paper we focus on interior optimum when the lower bound $U(0)$ does not bind; many arguments are simpler if the lower bound binds.

We start with two key optimality conditions. The first order conditions for u_v and w_v can be re-arranged as

$$\frac{\hat{\beta}_{t+1}}{\beta}\mathbb{E}_{\sigma_v}\left[k'_{t+1}\left(w_v\right)|z\right] = k'_t\left(v\right) \tag{26}$$

and

$$k'_t\left(v\right) = 1 - \zeta_t\mathbb{E}_{\sigma_v}\left[C'\left(u_v\right)|z\right]. \tag{27}$$

The first condition describes the law of motion for promised utilities. Since $\hat{\beta}_{t+1} \geq \beta$ with strict inequality when the sustainability constraint binds in period $t+1$, it shows a version of mean-reversion in promised utilities. Since k_t and k_{t+1} are concave functions with interior maxima, it implies that period $t+1$ promises have a drift towards the value that maximizes k_{t+1}. This drift component implies higher future promises for low-v agents and lower promises for high-v agents. This mean-reverting drift helps the government to relax sustainability constraints in future periods. The second condition provides a linkage between promised and current utility allocations.

For our purposes it is important to establish bounds for u_v and w_v. The optimality conditions (see Appendix for details) imply that

$$\left(1-k_t'(v)\right)\theta_1 \leq \zeta_t C'\left(u_v(m,z)\right) \leq \left(1-k_t'(v)\right)\theta_{|\Theta|}, \tag{28}$$

and

$$\underline{\varrho}\left[1-k_t'(v)\right]+\left(1-\frac{\beta}{\hat{\beta}_{t+1}}\right) \leq 1-k_{t+1}'\left(w_v(m,z)\right) \leq \bar{\varrho}\left[1-k_t'(v)\right]+\left(1-\frac{\beta}{\hat{\beta}_{t+1}}\right) \tag{29}$$

for some numbers $\bar{\varrho}$, $\underline{\varrho}$ with a property that $\bar{\varrho}, \underline{\varrho} \to \beta/\hat{\beta}_{t+1}$ as $\theta_{|\Theta|}-\theta_1 \to 0$.

Equation (28) shows the role of private information. If agents' types were observable, the planner would equalize the marginal costs of providing 1 util to type θ, $1-\frac{1}{\theta}\zeta_t C'(u)$, to the shadow cost of past promises, $k_t'(v)$, while allocating the same promised utility w to all types. This would give more consumption to higher θ types, which is not incentive compatible. With private information the dispersion of current period consumption allocations is smaller, which can be seen by the bounds (28). Equation (29) shows, in addition, how the sustainability constraints in future periods interact with the provision of incentives in the current period. It shows that bounds for future promises $1-k_{t+1}'(w_v)$ shrink as $v \to \underline{v}$. When future sustainability constraints bind, the government realizes that more dispersion of inequality tomorrow makes it harder to sustain efficient outcomes, which limits the dispersion of promises that the government gives and prevents the long run immiseration.

Conditions similar to (26)-(29) also appear in the problems in which the government can fully commit (see Atkeson and Lucas (1992) and Farhi and Werning (2007)). Our analysis generalizes those conditions to the environments with no commitment. A novel feature of such environments is that complete information revelation is generally suboptimal. We next turn to the characterization of the efficient information revelation.

Suppose that it is optimal for type θ to randomize between messages m' and m''. The optimality for the reporting strategy σ_v together with the envelope theorem yields[11]

$$\begin{aligned}
&\left(\theta u_v(m'',z)-\zeta_t C\left(u_v(m'',z)\right)+\hat{\beta}_{t+1}k_{t+1}\left(w_v(m'',z)\right)\right) \\
&-\left(\theta u_v(m',z)-\zeta_t C\left(u_v(m',z)\right)+\hat{\beta}_{t+1}k_{t+1}\left(w_v(m',z)\right)\right) \\
&=\chi_t\left\{\left[\theta u_v^w(m'',z)-\lambda_t^w C\left(u_v^w(m'',z)\right)\right]-\left[\theta u_v^w(m',z)-\lambda_t^w C\left(u_v^w(m',z)\right)\right]\right\}.
\end{aligned} \tag{30}$$

[11] Formally, we take a derivative of σ_v in the direction σ', defined as $\sigma'(m'|\theta,z)=\sigma_v(m'|\theta,z)+\sigma_v(m''|\theta,z)$, $\sigma'(m''|\theta,z)=0$ and $\sigma'(m|\theta,z)=\sigma_v(m|\theta,z)$ for all other m,θ, and apply (15).

Equation (30) captures benefits and costs of information revelation. The left hand side of the equation captures the benefits of better information revelation on the equilibrium path. It compares the "cost-weighted" utility that the government obtains for type θ if it receives messages m' and m''. This term is the general form of the expression $\kappa_0\left(v;\sigma^{in}\right)-\kappa_0\left(v;\sigma^{un}\right)$ that appeared in our simplified example in Section 4.1. The right hand side of the equation captures the cost of information revelation off the equilibrium path. If message m'' reveals more information about θ than m', in a sense that the expectation of θ conditional on m'' is further from θ than the expectation conditional on m', this expression is positive. The right hand side of (30) is the analogue of $\chi_0\left\{W_0\left(\sigma^{in}\right)-W_0\left(\sigma^{un}\right)\right\}$ in Section 4.1.

Equation (30) develops the key intuition for efficient information revelation. First, consider information revelation for low values of v. Bounds (28) and (29) imply the expression on the left hand side of (30) converges to zero as $v\to\underline{v}$, so that the gains from information revelation disappear for low values of v. Therefore, the optimal reporting strategy must converge to a strategy that minimizes the off the equilibrium path cost of information revelation. Such strategy must be uninformative by Lemma 2. Note that the uninformative strategy assigns the same posterior belief for any reported message, so that that right hand side of (30) is zero. In the Appendix we prove a stronger result that the uninformative strategy is optimal for all v sufficiently low.

Theorem 1 *Suppose that the sustainability constraint (10) binds in periods t and $t+1$. There exists $v_t^->\underline{v}$ such that for all $v\in\left[\underline{v},v_t^-\right]$ the allocation (u_v,w_v,σ_v) does not depend on z, strategy σ_v is uninformative and $(u_v(m),w_v(m))=(u_v(m'),w_v(m'))$ for all m,m'.*

We now turn to the analysis of information revelation for high values of v. Consider the case when $|\Theta|=2$. By Proposition 3 we can restrict attention to at most 3 messages, such that reporting two of those messages reveals the type of the sender perfectly. Therefore the left hand side of (30) compares on-the-equilibrium-path benefits from a message that reveals full information about a sender and a message that reveals only some of the information. Bounds (28) and (29) can be used to show that as long as the coefficient of absolute risk aversion, $U''(c)/U'(c)$ goes to zero for high c, the informational gains from better information must go to infinity as $v\to\bar{v}$. Since the informational costs are bounded by Lemma 2, this implies that equation (30) cannot hold and each agent must reveal his type fully for all v high enough.

Formally, suppose that utility satisfies

Assumption 1 ***(decreasing absolute risk aversion)*** *U is twice continuously differentiable* *and*

$$\lim_{c \to \infty} U''(c) / U(c) = 0. \tag{31}$$

We say that σ_v *reveals full information about type* θ_i if no type $j \neq i$ sends the same messages as type θ_i with positive probability.

Theorem 2 *Suppose that Assumption 1 is satisfied and* $|\Theta| = 2$*. Then there exists* $v_t^+ < \bar{v}$ *such that for all* $v \geq v_t^+$*,* σ_v *reveals full information about each type.*

Previous arguments can also be extended for the first and the last type of arbitrary message spaces.

Corollary 2 *Suppose that Assumption 1 is satisfied.*

1. ***(no informational distortions at the top)****. Then* σ_v *reveals full information about* θ_1 *for all* v *sufficiently high.*

2. ***(no informational distortions at the bottom)****. Suppose in addition that* $\underline{\varrho} \geq 0$ *and* $\pi\left(\theta_{|\Theta|-1}\right)\left(\theta_{|\Theta|} - \theta_{|\Theta|-1}\right) > \left(\pi\left(\theta_{|\Theta|-1}\right) + \pi\left(\theta_{|\Theta|}\right)\right)\left(\theta_{|\Theta|-1} - \theta_{|\Theta|-2}\right)$*. Then* σ_v *reveals full information about* $\theta_{|\Theta|}$ *for all* v *sufficiently high.*

The first part of this Corollary shows that full information for the lowest type θ_1 is optimal for high v. It is an informational analogue of the "no distortion at the top" result from the mechanism design literature (see, e.g. Mirrlees (1971)). In the mechanism design version of our environment, the incentive constraints bind upward, so that θ_1 is the "top type" in a sense that no other type wants to pretend to be θ_1. This property of the top types drives both the "no distortion at the top" result in standard mechanism design models with commitment and part 1 of Corollary 2 in our environment.

Part 2 of Corollary 2 shows that informational distortions are suboptimal for the highest $\theta_{|\Theta|}$ as long as the types are not too close to each other. The extra condition is needed to rule out bunching. When some types are close, it might be optimal to give them the same allocations even if they reveal their type fully (as, for example, in models with commitment) in order to provide incentives for other types. Since bunching is suboptimal with two types, no additional conditions were necessary in Theorem 2.

More insight into the behavior of the interior types can be gained from assuming that the utility function satisfies $U(c) = ac^{1/a}$ for $a > 1$. In this case it is easy to find the limiting allocations towards (u_v, w_v, σ_v) converge as $v \to \bar{v}$. In particular, we can use homogeneity of

C and boundedness from below of u to show that for any v, $k_t(v_0 v)/v^a \to \tilde{k}_t(v)$ as $v_0 \to \bar{v}$, where function $\tilde{k}_t(v)$ is defined by

$$\tilde{k}_t(v) = \max_{\substack{\{u(m),w(m),\sigma(m|\theta)\}_{m\in M_\Theta} \\ \sigma(\cdot|\theta)\in\Delta(M_\Theta)}} \mathbb{E}_\sigma\left[-\zeta_t C(u) + \hat{\beta}_{t+1} k_{t+1}(w)\right] \tag{32}$$

subject to (22) and (21). Moreover, σ_v converges to a solution $\tilde{\sigma}$ of this problem.

The limiting problem (32) has no cost of information revelation. It is equivalent to a standard mechanism design problem in which agents can send reports over a redundantly large message set M_Θ. If it is suboptimal to bunch different types when agents report over Θ, it is also suboptimal to bunch types when agents report over M_Θ. A sufficient condition to rule out bunching in mechanism design problems is an assumption that types are not too close, which we state as Assumption 2. It turns out to be a sufficient condition for full information revelation for high enough v in our economy without commitment.

Assumption 2 *The distribution of θ satisfies*

$$\pi(\theta_{n-1})[\theta_n - \theta_{n-1}] - (\theta_{n-1} - \theta_{n-2}) \sum_{i=n-1}^{|\Theta|} \pi(\theta_i) \geq 0 \text{ for all } n > 2. \tag{33}$$

Note that this assumption is satisfied both in Theorem 1 and Corollary 2. Suboptimality of bunching in the limit implies that information revelation is suboptimal for all sufficiently high v. The proof of this result is a tedious but mostly straightforward application of the Theorem of Maximum and we leave it for the Supplementary material.

Proposition 4 *Suppose that $U(c) = ac^{1/a}$ for $a > 1$ and Assumption 2 is satisfied. Then there exists $v_t^+ < \bar{v}$ such that for all $v \geq v_t^+$, σ_v reveals full information about each type.*

5 Extensions

We discussion two extensions of the baseline analysis.

5.1 Invariant distribution

In the previous sections we used arbitrary distributions of initial lifetime utilities ψ. In this section we briefly discuss two implications for our analysis if ψ is an invariant distribution. In this case multipliers $\hat{\beta}, \chi$ and ζ do not depend on t, with $\hat{\beta} > \beta$, $\chi > 0$ needed to prevent immiseration.[12]

[12] For given $\hat{\beta}, \chi, \zeta$ one can follow the arguments of Farhi and Werning (2007) to show that there exists initial distibution $\tilde{\psi}$ with a property that ψ_t generated by a solution to (20) is such that $\psi_t = \tilde{\psi}$ for all t. This distribution is invariant if feasibility and sustainabilty constraints hold with equalities. We leave it for future research to explore under which conditions these constraints are satisfied.

One implication of the invariant distribution is that no agent is stuck forever in the region in which no information is revealed. To see this, note that equation (29) shows that agent's promised utility strictly increases with positive probability if $v_t < v_*$ and strictly decreases with positive probability if $v_t > v_*$, where v_* is given by $k'(v_*) = 0$. Suppose there is no information revelation for all $v \leq v_* + a$ where $a \geq 0$. Then the invariant distribution must assign all the mass to v_* and, moreover, no information about agents is revealed. But in this case the sustainability constraint is slack, which is a contradiction. Therefore the "no information" region lies strictly below v_*.

Another implication of Theorem 1 and bounds (29) in this case is that there must exists an endogenous lower bound beyond which agents' lifetime utility never falls. Once an agent's utility reaches that bound, he bounces from it with a positive probability. This dynamics resembles that of Atkeson and Lucas (1995), except that in their case the utility bound was set exogenously. Near the endogenous lower bound agent may reveal no information and receive no insurance against that period's shocks.

5.2 Persistent shocks

We chose to focus on iid shocks in our benchmark analysis. Most of our key results extend with few modifications to Markov shocks. When the shocks are Markov, let $\pi\left(\theta|\theta^-\right)$ denote the probability of realization of shock θ conditional on shock θ^- in the previous period. We assume that $\pi\left(\theta|\theta^-\right) > 0$ for all θ, θ^-. Let $\pi^t\left(\theta|\theta^-\right)$ be the probability of realization of θ conditional on shock θ^- being realized t periods ago.

Many arguments in the persistent case are straightforward extentions of our previous analysis. We briefly sketch them in this section leaving the details for the Supplementary material. Following the same steps as before, we can show that in the worst equilibrium there is no information revelation for the government. The payoff in that equilibrium, unlike the iid case, depends on the prior beliefs that the government has. The maximum payoff that the government can achieve in any period t is given by

$$\tilde{W}_t\left(\mu_t\right) = \sup_{\{u_{t+s}(h)\}_{h\in H^t, s\geq 0}} \int_{H^t} \sum_{s=0}^{\infty} \beta^s \mathbb{E}_{\mathbf{p}_{t+s}}\left[\theta|h\right] u_{t+s}\left(h\right) d\mu_t$$

subject to the feasibility constraint (8) holding for all $t+s$ and

$$p_{t+s}\left(\theta|h^t\right) = \int_{\Theta} \pi^s\left(\theta|\theta^-\right) p_t\left(d\theta^-|h^t\right) \text{ for } s > 0.$$

Similarly to the iid case, we can bound $\tilde{W}_t\left(\mu_t\right)$ with a function that is linear in μ_{t-1}. Define the analogue of (13) as

$$W_t(\sigma, p) =$$

$$\max_{\{u_{t+s}(m)\}_{s\geq 0}} \int_{M_t\times\Theta\times\Theta} \sum_{s=0}^{\infty} \beta^s \left(\int_{\Theta} \pi^s(d\theta_s|\theta)\,\theta_s u_{t+s}(m) - \lambda^w_{t,t+s}\left(C\left(u_{t+s}(m)\right) - e\right) \right) \sigma(dm|\theta)\,\pi\left(d\theta|\theta^-\right) p\left(d\theta^-\right),$$

then Lemma 2, and in particular bound (14), can be extended to $W_t(\sigma_t, p_t)$. It is straightforward then to obtain the analogue of Lagrangian (17). This Lagrangian can then be rewritten recursively using the techniques of Fernandes and Phelan (2000), who studied an economy with persistent shocks and commitment. In their formulation the state is a vector $\mathbf{v} = \left(v\left(\theta_1^-\right), ..., v\left(\theta_{|\Theta|}^-\right)\right)$ of promised utilities and the realization of the shock in the previous period, θ^-. In our environment $\mathbf{v}$ remains a state variable with an analogue of the promise keeping constraints (21) holding for each type,

$$v\left(\theta_i^-\right) = \int \left[\theta u(m,z) + \beta w(m,z,\theta)\right] \sigma(dm|z,\theta)\,dz\,\pi\left(d\theta|\theta_i^-\right) \equiv \mathbb{E}_\sigma\left[\theta u + \beta w|\theta_i^-\right]. \qquad (34)$$

In the environments with commitment each agent plays a pure reporting strategy and θ^- is known along the equilibrium path. In our settings θ^- is not known, but the planner forms posterior beliefs about θ^-. Therefore, the only difference from the commitment environment is that the posterior belief p replaces θ^- as the state variable.

The recursive formulation is given by

$$k_t(\mathbf{v}, p) = \max_{\substack{(u,w,\sigma) \\ \sigma(\cdot|\theta)\in\Delta(M_\Theta),\, p'\in\Delta(\Theta)}} \sum_{\theta^-\in\Theta} p\left(\theta^-\right) \mathbb{E}_\sigma\left[\theta u - \zeta_t C(u) + \hat{\beta}_{t+1} k_{t+1}\left(\mathbf{w}, p'\right)|\theta^-\right] - \chi_t \int W_t dz \qquad (35)$$

subject to (34), the incentive constraint

$$\mathbb{E}_\sigma\left[\theta u + \beta w(\cdot,\cdot,\theta)|\theta, z\right] \geq \mathbb{E}_{\sigma'}\left[\theta u + \beta w(\cdot,\cdot,\theta)|\theta, z\right] \text{ for all } z, \theta, \sigma',$$

and the Bayes rule condition that requires p' to satisfy, whenever defined,[13]

$$p'(\theta|m,z) = \frac{\sigma(m|\theta)\sum_{\theta_-\in\Theta}\pi\left(\theta|\theta^-\right)p\left(\theta^-\right)}{\sum_{(\theta,\theta_-)\in\Theta^2}\sigma(m|\theta)\,\pi\left(\theta|\theta^-\right)p\left(\theta^-\right)}.$$

The analysis of dynamic mechanism design problems with persistent shocks is more involved since the recursive problem with additional state variables is more complicated. The broad lessons about desirability of information revelation for high and low promised values (now in the sense of increasing each element of vector $\mathbf{v}$) still holds. First of all, the example in

[13]Strictly speaking m should be replaces with any Borel set A of $\Delta(M_t)$ analogously to definition (3).

Section 4.1 remains virtually unchanged. More specifically, in the Supplementary material we follow similar steps as in the iid case and define functions $\kappa_1(\mathbf{v}, p)$ and $\kappa_0(\mathbf{v}; \sigma)$. We show that both these functions are homogenous in $\mathbf{v}$: for any $x > 0$, $\kappa_1(x\mathbf{v}) = x^a \kappa_1(\mathbf{v}, p)$ and $\kappa_0(x\mathbf{v}; \sigma) = x^a \kappa_0(\mathbf{v}; \sigma)$. Note that $\mathbf{v} \geq 0$ if $1/a \in (0, 1)$ and $\mathbf{v} < 0$ if $1/a < 0$. We can then conclude that no information revelation is optimal for low values of $x\mathbf{v}$ (i.e. $x \to 0$ if $1/a \in [0, 1)$ and $x \to \infty$ if $1/a < 0$) and, when no bunching is desirable with commitment, full information is optimal for high values of $x\mathbf{v}$. We can also generalize Theorem 1 to the persistent case (see the Supplementary material for details).

Proposition 5 *Suppose that utility is bounded. Let $\sigma_{\mathbf{v},p}$ be a solution to (35) and $\bar{\sigma}$ an uninformative strategy. If the sustainability constraint (10) binds in periods t, then* $\Pr(\sigma_{\mathbf{v},p} \to \bar{\sigma}) \to 1$ *as $\mathbf{v} \to \mathbf{0}$ for all p.*

6 Final remarks

In this paper we took a step towards developing of theory of social insurance in a setting in which the principal cannot commit. We focused on the simplest version of no commitment that involves a direct, one-shot communication between the principal and agents, and showed how such model can be incorporated into the standard recursive contracting framework with relatively few modifications. Our approach can be extended to other types of communication. For example, a literature starting with a seminar work of Myerson (1982) considered communication protocols that involve a mediator. Such protocols may further increase welfare (see, e.g. Bester and Strausz (2007)) but they require some ability to commit by the policy maker to use the mediating device. The key difference is that, after sending any message m, the agent receives a stochastic allocation, so that the incentive constraint (25) holds in expectation rather than message-by-message. Two main insights of the paper – recursive characterization and Theorems 1 and Theorems 2 – depend little on the detailed structure of the incentive constraints. We believe that our analysis can be extended to such settings in a relatively straightforward way and leave it to future research.

Another important direction that needs to be explored in future work is how the allocations in best equilibria can be decentralized through a system of taxes and transfers. Some of the most fruitful way to proceed may be to extend our basic set up to incorporate such margins as labor supply or job search effort, and consider decentralizations along the lines of labor taxation in Albanesi and Sleet (2006) and unemployment insurance in Atkeson and Lucas (1995).

7 Appendix

7.1 Proof of Lemma 2

First we show that sequences $\{\lambda_t^w\}_t$ is uniformly bounded away from 0. The objective function in (7) is concave and the constraint set is convex, therefore there exists $\lambda_t^w \geq 0$ such that the solution to (7), $\left\{\underline{u}^w\left(h^t\right)\right\}_{h^t \in H^t}$, satisfies $C'\left(\underline{u}^w\left(h^t\right)\right) = \frac{1}{\lambda_t^w}\mathbb{E}_{\sigma_t}\left[\theta|h^t\right]$. The Lagrange multiplier λ_t^w is given by

$$\int_{H^t} C\left(C'^{-1}\left(\frac{1}{\lambda_t^w}\mathbb{E}_{\sigma_t}\left[\theta|h\right]\right)\right)d\mu_t = e. \tag{36}$$

Since $\mathbb{E}_{\sigma_t}\left[\theta|h\right] \in \left[\theta_1, \theta_{|\Theta|}\right]$, the left hand side of (36) is continuous in λ_t^w and goes to 0 and infinity as λ_t^w goes infinity and 0 respectively. Therefore λ_t^w is bounded away from 0 from below and bounded above, and these bounds can be chosen independently of t. This also implies that the supremum to (7) is obtained. The same arguments applied to (13) establish that W_t is well defined, $u^w\left(m\right)$ satisfies

$$C'\left(u^w\left(m\right)\right) = \frac{1}{\lambda_t^w}\mathbb{E}_{\sigma}\left[\theta|m\right] \in \left[\frac{\theta_1}{\lambda_t^w}, \frac{\theta_{|\Theta|}}{\lambda_t^w}\right], \tag{37}$$

and family $\{W_t\}_t$ is uniformly bounded. Continuity of W_t then follows from the Theorem of Maximum since (37) implies that we can restrict $\{u\left(m\right)\}_{m\in M_t}$ in maximization problem (13) to a compact set.

To see that W_t is convex, for any σ', σ'' and $\alpha \in [0,1]$,

$$\begin{aligned}
W_t\left(\alpha\sigma' + (1-\alpha)\sigma''\right) &= \max_{\{u(m)\}_{m\in M_t}}\left\{\alpha\int_{M_t\times\Theta}\left[\theta u\left(m\right) - \lambda_t^w C\left(u(m)\right)\right]\sigma'\left(m|\theta\right)\pi\left(d\theta\right)\right. \\
&\quad \left. + (1-\alpha)\int_{M_t\times\Theta}\left[\theta u\left(m\right) - \lambda_t^w C\left(u(m)\right)\right]\sigma''\left(dm|\theta\right)\pi\left(d\theta\right)\right\} + \lambda_t^w e \\
&\leq \alpha\max_{\{u(m)\}_{m\in M_t}}\left\{\int_{M_t\times\Theta}\left[\theta u\left(m\right) - \lambda_t^w C\left(u(m)\right)\right]\sigma'\left(dm|\theta\right)\pi\left(d\theta\right)\right\} \\
&\quad + (1-\alpha)\max_{\{u(m)\}_{m\in M_t}}\left\{\int_{M_t\times\Theta}\left[\theta u\left(m\right) - \lambda_t^w C\left(u(m)\right)\right]\sigma''\left(dm|\theta\right)\pi\left(d\theta\right)\right\} + \lambda_t^w e \\
&= \alpha W_t\left(\sigma'\right) + (1-\alpha)W_t\left(\sigma''\right).
\end{aligned}$$

To see that W_t is minimized if and only if a strategy is uninformative, consider any uninformative $\bar{\sigma}$. From (37), the solution to (13) given $\bar{\sigma}$ satisfies $C'\left(\bar{u}^w\left(m\right)\right) = \frac{1}{\lambda_t^w}$ a.s. and hence $W_t\left(\bar{\sigma}\right) = C'^{-1}\left(\frac{1}{\lambda_t^w}\right) - \lambda_t^w C\left(C'^{-1}\left(\frac{1}{\lambda_t^w}\right)\right)$. Let $\mu\left(A\right) = \int_\Theta \sigma\left(A|\theta\right)\pi\left(d\theta\right)$ for any Borel A of message set M and note that for any σ

$$\begin{aligned} & W_t(\sigma) - W_t(\bar{\sigma}) \\ = & \max_{\{u(m)\}_{m\in M_t}} \int_{M_t\times\Theta} \left[(\theta u(m) - \lambda_t^w C(u(m))) - (\theta\bar{u}^w - \lambda_t^w C(\bar{u}^w))\right] \sigma(dm|\theta)\,\pi(d\theta) \\ = & \max_{\{u(m)\}_{m\in M_t}} \int_{M_t} \left\{(\mathbb{E}_\sigma[\theta|m]\,u(m) - \lambda_t^w C(u(m))) - (\mathbb{E}_\sigma[\theta|m]\,\bar{u}^w - \lambda_t^w C(\bar{u}^w))\right\} \mu(dm). \end{aligned}$$

The expression in curly bracket is non-negative, which implies that $W_t(\sigma) \geq W_t(\bar{\sigma})$ for all σ. Moreover for informative σ, there is a set of messages A with $\mu(A) > 0$ such that $|\mathbb{E}_\sigma[\theta|A] - 1| > 0$. For all such messages the expression in curly brackets is strictly positive since $\bar{u}^w$ does not satisfy the optimality condition (37) for $m \in A$. Since $\mu(A) > 0$, the expression above is strictly positive therefore any informative σ cannot be a minimum.

We prove (15) by using Theorem 3 in Milgrom and Segal (2002). For completeness, we state it here adapted to our set up

Theorem 3 *Let $V(t) = \sup_{x\in X} f(x,t)$ where $t \in [0,1]$, and let $x^*(t)$ be a solution to this problem given t and f_t be the derivative of f with respect to t. Suppose that (i) for any t a solution $x^*(t)$ exists, (ii) $\{f_t(x,\cdot)\}_{x\in X}$ is equicontinuous, (iii) $\sup_{x\in X}|f(x,t)| < \infty$ at $t = t_0$, and (iv) $f_t(x^*(t), t_0)$ is continuous in t at $t = t_0$. Then $V'(t_0)$ exists and $V'(t_0) = f_t(x^*(t_0), t_0)$.*

In our case $u = \{u(m)\}_{m\in M_t}$ can be restricted to a compact set and

$$\int_{M_t\times\Theta} (\theta u(m) - \lambda_t^w C(u(m)))\left[(1-\alpha)\sigma(m|\theta) + \alpha\sigma'(m|\theta)\right] dm\pi(d\theta)$$

is continuous in $\{u(m)\}_{m\in M_t}$, hence conditions (i) and (iii) of this theorem are satisfied. The derivative with respect to α is

$$f_\alpha(\{u\},\alpha) \equiv \int_{M_t\times\Theta} (\theta u(m) - \lambda_t^w C(u(m)))\left[\sigma'(m|\theta) - \sigma(m|\theta)\right] dm\pi(d\theta),$$

and it does not depend on α, therefore $\{f_\alpha(\{u\},\cdot)\}_{\{u\}}$ is equicontinuous. For any $\alpha \in [0,1]$, the solution to (13), $u^w(\alpha)$, given $(1-\alpha)\sigma + \alpha\sigma'$ is unique and, by the Theorem of Maximum, continuous in α. Therefore, $f_\alpha(\{u(\alpha)\}, \alpha_0)$ is continuous in α at $\alpha = \alpha_0$, which verifies condition (iv).

Since sequence $\{\lambda_t\}_t$ is uniformly bounded (away from zero from below), equation (37) implies that $u^w(m)$ is bounded and those bound do not depend on t. Therefore all families of directional derivatives $\{\partial W_t(\sigma)/\partial\sigma'\}_t$ are uniformly bounded.

We prove (14) next. By Lagrange Duality theorem (Luenberger (1969), Theorem 1, p. 224), $\tilde{W}_t(\mu_t)$ can be written as a minmax problem

$$\begin{aligned}
\tilde{W}_t(\mu_t) &= \min_{\lambda\geq 0}\max_{\{u(h)\}_{h\in H^t}}\int_{H^t}\left(\mathbb{E}_{\sigma_t}[\theta|h]\,u(h)-\lambda C(u(h))\right)d\mu_t+\lambda e \\
&\leq \max_{\{u(h)\}_{h\in H^t}}\int_{H^t}\left(\mathbb{E}_{\sigma_t}[\theta|h]\,u(h)-\lambda_t^w C(u(h))\right)d\mu_t+\lambda_t^w e \\
&= \max_{\{u(h)\}_{h\in H^t}}\int_{\breve{H}^{t-1}}\left[\int_{M_t\times\Theta}\left\{\theta u\left(\breve{h}^{t-1},m\right)-\lambda_t^w C\left(u(\breve{h}^{t-1},m)\right)\right\}\sigma_t\left(dm|\breve{h}^{t-1},\theta\right)\pi(d\theta)\right]dzd\mu_{t-1} \\
&= \int_{\breve{H}^{t-1}}W_t\left(\sigma_t\left(\cdot|\breve{h}^{t-1},\cdot\right)\right)dzd\mu_{t-1}.
\end{aligned}$$

where the inequality follows from the fact that λ_t^w may not be a minimizer for an arbitrary μ_t. By definition of λ_t^w this inequality becomes equality at μ_t^*, which establishes (14).

7.2 Proof of Proposition 3

First we prove some preliminary results. Fix z and consider any incentive compatible allocation that satisfies (22). Let $M_t(\theta)$ be as defined in the text. For any i let $M_t^-(\theta_i)=M_t(\theta_i)\cap M_t(\theta_{i-1})$, $M_t^+(\theta_i)=M_t(\theta_i)\cap M_t(\theta_{i+1})$ and $M_t^{excl}(\theta_i)=M_t(\theta_i)\setminus(\cup_{j\neq i}M_t(\theta_j))$.

Lemma 3 *For all i, $M_t(\theta_i)=M_t^-(\theta_i)\cup M_t^{excl}(\theta_i)\cup M_t^+(\theta_i)$.*

Proof. Suppose $m\in M_t(\theta_i)\setminus M_t^{excl}(\theta_i)$. Then there exists j such that $m\in M_t(\theta_j)$. Without loss of generality let $\theta_j>\theta_i$. For any $m'\in M_t(\theta_{i+1})$

$$\theta_{i+1}u\left(m'\right)+\beta w\left(m'\right)\geq\theta_{i+1}u(m)+\beta w(m)$$

and

$$\begin{aligned}
\theta_i u(m)+\beta w(m) &\geq \theta_i u\left(m'\right)+\beta w\left(m'\right), \\
\theta_j u(m)+\beta w(m) &\geq \theta_j u\left(m'\right)+\beta w\left(m'\right).
\end{aligned}$$

The sum of the first the second inequalities implies $u(m')\geq u(m)$, the sum of the first and the third inequalities implies $u(m')\leq u(m)$, therefore $u(m')=u(m)$ and $m\in M_t(\theta_{i+1})$. ■

Lemma 4 *Let (u^*,w^*,σ^*) be a solution to (20) for a given z. Then $(u^*(m'),w^*(m'))=(u^*(m''),w^*(m''))$ for all $m',m''\in M_t^-(\theta_t)$, for all $m',m''\in M_t^+(\theta_t)$ and for almost all (with respect to measure σ^*) $m',m''\in M_t^{excl}(\theta_i)$.*

Proof. The first part of the lemma follows immediately from the discussion in the text, so we prove the second part. Let $\hat{u} = \mathbb{E}_\sigma \left[u^*(m) | m \in M_t^{excl}(\theta)\right]$, $\hat{w} = \mathbb{E}_\sigma \left[w^*(m) | m \in M_t^{excl}(\theta)\right]$ and consider an alternative (u', w', σ^*) defined as $(u'(m), w'(m)) = (u^*(m), w^*(m))$ if $m \notin M_t^{excl}(\theta)$, and $(u'(m), w'(m)) = (\hat{u}, \hat{w})$ if $m \in M_t^{excl}(\theta)$. For any $\theta' \neq \theta$, $m' \in M_t(\theta')$

$$\theta' u^*(m') + \beta w^*(m') > \theta' u^*(m) + \beta w^*(m) \text{ for all } m \in M_t^{excl}(\theta),$$

therefore $\theta' u^*(m') + \beta w^*(m') > \theta' \hat{u} + \beta \hat{w}$, which implies that (u', w', σ^*) satisfies (22). It also satisfies (21) by construction, since θ gets the same utility from reporting any message $m \in M_t^{excl}(\theta)$. The objective function $\mathbb{E}_{\sigma^*}\left[\theta u - \zeta_t C(u) + \hat{\beta}_{t+1} k_{t+1}(w)\right]$ is strictly convex in (u, w), and would take a strictly higher value at (u', w') than at (u^*, w^*) unless (u^*, w^*) is constant on $M_t^{excl}(\theta_i)$ a.s. ■

Lemma 5 *Let (u^*, w^*, σ^*) be a solution to (20) for a given z. Suppose there exists a set $\tilde{M} \subset M_t$ s.t. $\sum_\theta \sigma^*\left(\tilde{M}|\theta\right)\pi(\theta) > 0$ and $u^*(m') = u^*(m'')$ for almost all (with respect to measure σ^*) $m', m'' \in \tilde{M}$. If $\chi_t > 0$ then $\mathbb{E}_{\sigma^*}[\theta|m'] = \mathbb{E}_{\sigma^*}[\theta|m'']$ a.s.*

Proof. Fix any $\hat{m} \in \tilde{M}$ and consider an alternative strategy σ' defined as $\sigma'(m|\theta) = \sigma^*(m|\theta)$ if $m \notin \tilde{M}$ and $\sigma'(\hat{m}|\theta) = \sigma^*\left(\tilde{M}|\theta\right)$ for all θ. Any agent reports any subset of messages with a positive probability with reporting strategy σ' only if he reports that subset with a positive probability with σ^*. Therefore the strategy profile $\sigma_\alpha = (1-\alpha)\sigma^* + \alpha\sigma'$ satisfies (21) and (22) for any $\alpha \in [0, 1]$. Let

$$f_{\hat{m}}(\alpha) \equiv \mathbb{E}_{\sigma_\alpha}\left[\theta u^* - \zeta_t C(u^*) + \hat{\beta}_{t+1} k_{t+1}(w^*)\right] - \chi_t W_t(\sigma_\alpha).$$

Since $u^*(m') = u^*(m'')$, $w^*(m') = w^*(m'')$,

$$\begin{aligned} f'_{\hat{m}}(\alpha)|_{\alpha=0} &= -\chi_t \frac{\partial W_t(\sigma^*)}{\partial \alpha} \\ &= -\chi_t\{(\mathbb{E}_{\sigma'}[\theta|\hat{m}]\, u^w(\hat{m}) - \lambda_t^w C(u^w(\hat{m})))\Pr(m \in \tilde{M}) \\ &\quad - \int_{\tilde{M}\times\Theta} (\theta u^w(m) - \lambda_t^w C(u^w(m)))\sigma^*(m|\theta) dm \pi(d\theta)\}. \end{aligned}$$

Optimality of $\sigma^* = \sigma_0$ implies that $f'_{\hat{m}}(\alpha)|_{\alpha=0} \leq 0$ for all $\hat{m}$, so that it can be written as

$$\mathbb{E}_{\sigma^*}\left[\theta|\tilde{M}\right] u^w(\hat{m}) - \lambda_t^w C(u^w(\hat{m})) \geq \mathbb{E}_{\sigma^*}\left[\mathbb{E}_{\sigma^*}[\theta|m]\, u^w(m) - \lambda_t^w C(u^w(m))|\tilde{M}\right] \text{ all } \hat{m}.$$

Taking expectations conditional on $\tilde{M}$ of both sides, we obtain $\mathbb{E}_{\sigma^*}\left[\theta|\tilde{M}\right]\mathbb{E}_{\sigma^*}\left[u^w(\hat{m})|\tilde{M}\right] \geq \mathbb{E}_{\sigma^*}\left[\mathbb{E}_{\sigma^*}[\theta|m]\, u^w(m)|\tilde{M}\right]$ which implies that *cov* $(\mathbb{E}_{\sigma^*}[\theta|m], u^w(m)) \leq 0$. On the other hand, optimality (37) implies that $u^w(m)$ is monotonically increasing in $\mathbb{E}_{\sigma^*}[\theta|m]$, and therefore

$cov\left(\mathbb{E}_{\sigma^*}\left[\theta|m\right], u^w(m)\right) \geq 0$. The two conditions can be satisfied only if $\mathbb{E}_{\sigma^*}\left[\theta|m'\right] = \mathbb{E}_{\sigma^*}\left[\theta|m''\right]$ a.s. ■

Proof of Proposition 3 and Corollary 1. Lemmas 4 and 5 show that for a given z, any solution to (20) can take at most $2|\Theta| - 1$ distinct values of $\left(u^*\left(m\right), w^*\left(m\right), \mathbb{E}_{\sigma^*}\left[\theta|m\right]\right)$ except for possible values that occur with probability 0 under reporting strategy σ^* (Lemma 5 is stated for the case $\chi_t > 0$, but when $\chi_t = 0$ the problem is isomorphic to the standard mechanism design problem and usual arguments apply to show that $|\Theta|$ distinct allocations represent a solution). The outcomes that occur with probability zero can be replaced with any outcome from that set of $2|\Theta| - 1$ distinct values. Therefore without loss of generality the set M can be restricted to at most $2|\Theta| - 1$ messages in (20), and hence, by Proposition 2, to $\mathbf{M}$ in the modified problem (17).

It remains to verify that it is a best equilibrium since not every solution to (17) needs to be a PBE. Let $(\boldsymbol{\sigma}^*, \boldsymbol{\sigma}_G^*, \mathbf{p}^*)$ be a best PBE that generates $(\mathbf{u}^*, \mathbf{M}^*)$ along the equilibrium path. By the arguments leading to construction of problem (17), $(\mathbf{u}^*, \boldsymbol{\sigma}^*)$ is a solution to that problem given $\mathbf{M}^*$. It is easy to show that any equilibrium is payoff-equivalent to one in which the allocations in period t depends only on the expected utility $w_t^*\left(h^{t-1}\right) \equiv \mathbb{E}_\sigma\left[\sum_{s=t}^{\infty}\beta^{s-t}u_s^*|h^{t-1}\right]$ but not on the particular realizations of shocks in the previous $t-1$ periods (Lemma 9 in the Supplementary materials) and so we assume that $(\mathbf{u}^*, \boldsymbol{\sigma}^*)$ satisfies this property. Previous arguments imply that after any history $\breve{h}^t$ there can be at most $2|\Theta| - 1$ distinct values of $\left\{\left(u_t^*\left(\breve{h}^t, m_t\right), w_t^*\left(\breve{h}^t, m_t\right), \mathbb{E}_{\boldsymbol{\sigma}^*}\left[\theta|\breve{h}^t, m_t\right]\right)\right\}_{m_t}$ that occur with probability 1 given $\left\{\sigma_t^*\left(\cdot|\breve{h}^t, \theta\right)\right\}_{\theta\in\Theta}$. Therefore it is possible to construct another strategy $\boldsymbol{\sigma'}$ which is incentive compatible and delivers the same payoff to the agents as $\boldsymbol{\sigma}^*$. The pair $(\boldsymbol{\sigma'}, \sigma_G^*)$ generates a distribution of histories $\boldsymbol{\mu'}$ with a property that $\Pr_{\boldsymbol{\mu'}}\left((u_t, \mathbb{E}_\sigma\theta_t) \in A\right) = \Pr_{\boldsymbol{\mu}^*}\left((u_t, \mathbb{E}_\sigma\theta_t) \in A\right)$ for any Borel set A of $\Delta\left(\left[(1-\beta)\underline{v}, (1-\beta)\bar{v}\right] \times \Theta\right)$. Therefore $\tilde{W}_t\left(\mu_t'\right) = \tilde{W}_t\left(\mu_t^*\right)$ and $\mathbb{E}_{\boldsymbol{\sigma'}}C\left(u_t\right) = \mathbb{E}_{\boldsymbol{\sigma}^*}C\left(u_t\right)$, which verifies that $(\boldsymbol{\sigma'}, \boldsymbol{\sigma}_G^*, \mathbf{p}^*)$ is a best PBE and proves Proposition 3**.**

If removing all non-distinct messages leaves fewer than $|M_\Theta|$ messages, extra messages can be added and agents strategies extended to keep posterior probabilities for such messages the same as for some of the original messages. By appropriately choosing which messages to add it is possible construct strategies that satisfy (25). This argument extends for all histories, and hence any value that $k_0\left(v\right)$ achieves given any $\mathbf{M}$, can also be achieved given $M_t = M_\Theta$ for all t. This completes the proof of Corollary 1. ■

7.3 Proofs for Section 4

First, we use Proposition 3 to simplify our problem. We drop explicit dependence of all variables on z and throughout this section all the results are understood to hold for all z. By Proposition 3 we can restrict attention to messages in a set M_v of cardinality no greater than $2|\Theta|-1$ such that no two distinct reports give the agent the same allocation. We use N_v to denote cardinality of M_v and $M_v(\theta)$ to denote the set of messages in M_v that maximize agent θ's utility. The incentive constraints (22) can be written as

$$\theta u(m_\theta)+\beta w(m_\theta) \geq \theta u(m')+\beta w(m'), \tag{38}$$
$$\sigma(m'|\theta)\left[(\theta u(m_\theta)+\beta w(m_\theta))-(\theta u(m')+\beta w(m'))\right]=0 \;\; \forall\, \theta\in\Theta, m_\theta \in M_v(\theta), m'\in M_v.$$

The triple (u_v, w_v, σ_v) is a solution to

$$\max_{u,w,\sigma} \mathbb{E}_\sigma\left[(1-\gamma_v)\theta u-\zeta_t C(u)+\hat{\beta}_{t+1}k_{t+1}(w)-\gamma_v\beta w\right]-\chi_t W_t(\sigma) \tag{39}$$

subject to (38) where $\gamma_v = k_t'(v)$.[14] Without loss of generality we arrange messages in M_v so that $u_v(m_1) < ... < u_v(m_{N_v})$, which also implies that $w_v(m_1) > ... > w_v(m_{N_v})$. Let $\xi'(\theta, m_\theta, m')$ and $\xi''(\theta, m_\theta, m')$ be Lagrange multipliers on the first and second constraint (38). We set $\xi(\theta, m, m')=0$ for all $m\notin M_v(\theta)$ and $\xi(\theta,m,m)=0$ for all m, so that ξ', ξ'' are well defined for all (θ, m, m').

The next lemma establishes bounds on optimal allocations that are important for establishing our main results in this section.

Lemma 6 *(u_v, w_v, σ_v) satisfies (26). For all v such that $k_t'(v)\leq 1$, equations (27), (28) and (29) hold, with* $\bar{\varrho}=\frac{\beta}{\hat{\beta}_{t+1}}\frac{1+\theta_{|\Theta|}-\theta_1}{\theta_1}$, $\underline{\varrho}=\frac{\beta}{\hat{\beta}_{t+1}}\frac{\theta_1-\theta_{|\Theta|}+1}{\theta_{|\Theta|}}$.

Proof. The first order conditions for $w_v(m)$ are

$$\sum_{\theta\in\Theta}\left[\frac{\hat{\beta}_{t+1}}{\beta}k_{t+1}'(w_v(m))-\gamma_v\right]\sigma_v(m|\theta)\pi(\theta)+\sum_{(\theta,m')\in\Theta\times M_v}\left[\xi'(\theta,m,m')+\sigma_v(m|\theta)\xi''(\theta,m,m')\right]$$
$$-\sum_{(\theta,m')\in\Theta\times M_v}\left[\xi'(\theta,m',m)+\sigma_v(m'|\theta)\xi'(\theta,m',m)\right]=0. \tag{40}$$

Sum over m to obtain

$$\sum_{(\theta,m)\in\Theta\times M_v}\left[\frac{\hat{\beta}_{t+1}}{\beta}k_{t+1}'(w_v(m))\right]\sigma_v(m|\theta)\pi(\theta)=\gamma_v=k'(v),$$

[14] To see this, let γ_v be the Lagrange multiplier on the promise keeping constraint in the original, convexified Bellman equation. By Proposition 1 and the envelope theorem it satisfies $\gamma_v = k_t'(v)$. Form a Lagrangian with γ_v and observe that it can be maximized for each z separately, which yields (39).

which is expression (26).

For what follows, it is useful to establish bounds on the Lagrange multipliers. Consider a message m_1. If $N_v = 1$, then trivially ξ' and ξ'' are two scalars equal to zero. Suppose $N_v > 1$ and let θ' be the largest θ such that $m_1 \in M_v(\theta)$. Since sending message m_1 is strictly suboptimal for any $\theta > \theta'$, $\xi'(\theta, m', m_1) = \sigma_v(m_1|\theta)\,\xi''(\theta, m', m_1) = 0$ for all (θ, m') with $\theta > \theta'$. Moreover, $\theta' u_v(m_1) + \beta w_v(m_1) \geq \theta' u_v(m) + \beta w_v(m)$ and $u_v(m_1) < u_v(m)$ for all $m \neq m_1$, which implies $\theta u(m_1) + \beta w(m_1) > \theta u(m) + \beta w(m)$ for all $m \neq m_1$ if $\theta < \theta'$. Therefore, $\xi'(\theta, m_1, m') = \sigma_v(m'|\theta)\,\xi''(\theta, m_1, m') = 0$ for all (θ, m') with $\theta < \theta'$. Thus we can write (40) for m_1 as

$$-\left(\frac{\hat{\beta}_{t+1}}{\beta} k'_{t+1}(w_v(m_1)) - \gamma_v\right) = \vartheta(m_1), \tag{41}$$

where

$$\vartheta(m_1) = \frac{\sum_{m' \in M_v}\left[\xi'\left(\theta', m_1, m'\right) + \sigma_v\left(m'|\theta'\right)\xi''\left(\theta', m_1, m'\right)\right] - \left[\xi'\left(\theta', m', m_1\right) + \sigma_v\left(m_1|\theta'\right)\xi''\left(\theta', m', m_1\right)\right]}{\sum_{\theta \in \Theta}\sigma_v(m_1|\theta)\,\pi(\theta)}.$$

Since w is decreasing in m and k_{t+1} is concave, $\frac{\hat{\beta}_{t+1}}{\beta} k'_{t+1}(w_v(m_1)) \leq \frac{\hat{\beta}_{t+1}}{\beta}\mathbb{E}_{\sigma_v}\left[k'_{t+1}(w_v)\right] = \gamma_v$, which implies that

$$\vartheta(m_1) \geq 0. \tag{42}$$

Similar steps for m_{N_v} establish $\vartheta(m_{N_v}) \leq 0$.

The first order conditions for $u_v(m)$ are

$$\sum_{\theta \in \Theta}\left[(1-\gamma_v)\,\theta - \zeta_t C'(u_v(m))\right]\sigma_v(m|\theta)\,\pi(\theta) + \sum_{(\theta, m') \in \Theta \times M_v}\left[\xi'\left(\theta, m, m'\right) + \sigma_v(m|\theta)\,\xi''\left(\theta, m, m'\right)\right]\theta$$
$$- \sum_{(\theta, m') \in \Theta \times M_v}\left[\xi'\left(\theta, m', m\right) + \sigma_v\left(m'|\theta\right)\xi''\left(\theta, m', m\right)\right]\theta = -\nu(m), \tag{43}$$

where $\nu(m) \geq 0$ is the Lagrange multipliers on the constraint $u(m) \geq U(0)$. It is zero if utility is unbounded below. Moreover, since we assumed $u_v(m_1) < \ldots < u_v(m_{N_v})$, it can only be strictly positive for m_1 even in the case when utility is bounded below.

We guess and verify that if $k'_t(v) = \gamma_v \leq 1$, then $\nu(m_1) = 0$.[15] Suppose $v(m_1) = 0$. Then the first order condition for $u_v(m_1)$ becomes

$$-\sum_{\theta \in \Theta}\left[(1-\gamma_v)\,\theta - \zeta_t C'(u_v(m_1))\right]\sigma_v(m_1|\theta)\,\pi(\theta) = \vartheta(m_1)\left(\sum_{\theta \in \Theta}\sigma_v(m_1|\theta)\,\pi(\theta)\right)\theta'. \tag{44}$$

[15] More precisely, for a fixed σ_v, maximization problem (39) is convex and thus necessary conditions are also sufficient.

Combining it with (42), it implies that

$$\zeta_t C'\left(u_v\left(m_1\right)\right) \geq \left(1-\gamma_v\right) \mathbb{E}_{\sigma_v}\left[\theta | m_1\right] \geq 0$$

and verifies that the lower bound for $u_v\left(m_1\right)$ does not bind. Analogous arguments establish

$$\left(1-\gamma_v\right) \mathbb{E}_{\sigma_v}\left[\theta | m_{N_v}\right] \geq \zeta_t C'\left(u_v\left(m_{N_v}\right)\right),$$

which, together with the monotonicity assumption on $u_v(m)$, gives (28).

For the rest of the proof consider v that satisfies $k_t'(v) \leq 1$. Summing (43) over m when $\nu(m)=0$ implies (27). Finally, combining (41) with (44) we obtain

$$\begin{aligned}
\frac{\hat{\beta}_{t+1}}{\beta} k_{t+1}'\left(w_v\left(m_1\right)\right) &= \gamma_v+\frac{1-\gamma_v}{\theta'} \mathbb{E}_{\sigma_v}\left[\theta | m_1\right]-\frac{1}{\theta'}+\frac{1}{\theta'}\left(1-\zeta_t C'\left(u_v\left(m_1\right)\right)\right) \\
&\geq \gamma_v+\frac{1-\gamma_v}{\theta'} \mathbb{E}_{\sigma_v}\left[\theta | m_1\right]-\frac{1}{\theta'}+\frac{1}{\theta'} \gamma_v \\
&\geq \frac{1-\gamma_v}{\theta'}\left(\theta_1-1\right)+\gamma_v,
\end{aligned}$$

where we used $\zeta_t C'\left(u_v\left(m_1\right)\right) \leq \zeta_t \mathbb{E}_{\sigma_v}\left[C'\left(u_v\right)\right]=1-\gamma_v$ to get the second inequality. Rearrange it to get

$$\begin{aligned}
1-k_{t+1}'\left(w_v\left(m_1\right)\right) &\leq \frac{\beta}{\hat{\beta}_{t+1}} \frac{\theta'-\theta_1+1}{\theta'}\left(1-\gamma_v\right)+\left(1-\frac{\beta}{\hat{\beta}_{t+1}}\right) \\
&\leq \frac{\beta}{\hat{\beta}_{t+1}} \frac{\theta_{|\Theta|}-\theta_1+1}{\theta_1}\left(1-\gamma_v\right)+\left(1-\frac{\beta}{\hat{\beta}_{t+1}}\right).
\end{aligned}$$

This, together with monotonicity of $w_v(m)$, establishes the upper bound in (29). Analogous manipulation of the first order condition for m_{N_v} establishes the lower bound in (29). ■

We use this lemma to establish bounds that we use to characterize information revelation for low v.

Lemma 7 *(a) If utility is unbounded below and constraint (10) binds in period* $t+1$, *then there are scalars* $A_{u,t}>0$, $A_{w,t}>0$ *and* v_t^- *such that for all* $v \leq v_t^-$ *any solution to (39) satisfies*

$$\begin{aligned}
0 &\leq \left|u_v(m)-u_v\left(m'\right)\right| \leq A_{u,t} \text{ for all } m, m', \\
0 &\leq \left|w_v(m)-w_v\left(m'\right)\right| \leq A_{w,t} \text{ for all } m, m'.
\end{aligned}$$

(b) If utility is bounded below, without loss of generality by zero, then $\lim_{v \rightarrow \underline{v}} u_v(m)=0$ *and* $\lim_{v \rightarrow \underline{v}} w_v(m)=0$ *for all* m.

Proof. (a) When (10) binds in $t+1$, $\hat{\beta}_{t+1} > \beta$ and therefore $\left(1 - \frac{\beta}{\hat{\beta}_{t+1}}\right) > 0$. When utility is unbounded below, $\lim_{v \to -\infty} k_t'(v) = 1$ and therefore expression (29) implies that there exists v_t^- such that

$$\frac{1}{2}\left(1 - \frac{\beta}{\hat{\beta}_{t+1}}\right) \leq 1 - k_{t+1}'(w_v(m)) \leq \frac{3}{2}\left(1 - \frac{\beta}{\hat{\beta}_{t+1}}\right) \text{ for all } m, v \leq v_t^-.$$

This establishes bounds for $w_v(m)$. The incentive constraint

$$\theta_1 u_v(m_1) + \beta w_v(m_1) \geq \theta_1 u_v(m_{N_v}) + \beta w_v(m_{N_v})$$

together with monotonicity $w_v(m_1) > ... > w_v(m_{N_v})$, $u_v(m_1) < ... < u_v(m_{N_v})$ imply that

$$\frac{\beta}{\theta_1}(w_v(m_1) - w(m_{N_v})) \geq u_v(m_{N_v}) - u_v(m_1) \geq 0$$

establishing bounds for u_v.

(b) When utility is bounded, Proposition 1 shows that $\lim_{v \to \underline{v}} k_t'(v) = \infty$. For any κ find the largest v^- that satisfy $k_t'(v^-) \geq \kappa$. By choosing arbitrarily high κ we can set v^- arbitrarily close to $\underline{v} = 0$. Let $\tilde{v} = \mathbb{E}_{\sigma_v}[\theta u_v + \beta w_v]$. Due to the possibility of randomization, in general $\tilde{v} \neq v$ but $k_t'(v) = k_t'(\tilde{v})$. Therefore $\tilde{v} \leq v^-$ if $v \leq v^-$. Then

$$\begin{aligned} v^- &\geq \tilde{v} = \mathbb{E}_{\sigma_v}[\theta u_v + \beta w_v] \\ &\geq \mathbb{E}_{\sigma_v}[\theta_1 u_v + \beta w_v] \\ &\geq \pi(\theta_1) \sum_{m \in M_v(\theta_1)} \sigma(m|\theta_1)[\theta_1 u_v(m) + \beta w_v(m)] \\ &\geq \pi(\theta_1)[\theta_1 u_v(m_1) + \beta w_v(m_1)] \\ &\geq \pi(\theta_1)\beta w_v(m_1) \geq 0. \end{aligned} \tag{45}$$

Here the second, third and fifth lines follows from nonnegativity of u and w, and the fourth line follows from the fact that if $\sigma(m|\theta_1) > 0$, then $(u_v(m), w_v(m))$ gives the same utility to θ_1 as $(u_v(m_1), w_v(m_1))$. Since v^- can be chosen to be arbitrarily close to 0, this implies that $\lim_{v \to \underline{v}} w_v(m_1) = 0$. Since $w_v(m_1) \geq w_v(m)$, this implies that $\lim_{v \to \underline{v}} w_v(m) = 0$ for all m. Analogous arguments show that $\lim_{v \to \underline{v}} u_v(m) = 0$. ■

7.3.1 Proofs for low v

We now can prove the first limiting result about optimal information revelation.

Lemma 8 *Suppose that sustainability constraint (10) binds in periods t and $t+1$. Then σ_v converges to an uninformed strategy as $v \to \underline{v}$.*

Proof. Let $\bar{\sigma}$ be any uninformative strategy. Let $\bar{u}_v(m) = \mathbb{E}_{\sigma_v}\theta u_v$, $\bar{w}_v(m) = \mathbb{E}_{\sigma_v} w_v$ for all m. Note that since $u_v(m)$ is increasing in m

$$u_v(m_1) = u_v(m_1)\,\mathbb{E}_{\sigma_v}\theta \le \bar{u}_v(m) \le u_v\left(m_{|\Theta|}\right)\mathbb{E}_{\sigma_v}\theta = u_v\left(m_{|\Theta|}\right),$$

and an analogues relationship holds for $\bar{w}_v$.

Profile $(\bar{u}_v, \bar{w}_v, \bar{\sigma})$ is incentive compatible and satisfies

$$\mathbb{E}_{\sigma_v}\left[\theta u_v + \beta w_v\right] = \mathbb{E}_{\bar{\sigma}}\left[\theta \bar{u}_v + \beta \bar{w}_v\right]. \tag{46}$$

The value of the objective function (39) evaluated as (u_v, w_v, σ_v) should be higher than evaluated at $(\bar{u}_v, \bar{w}_v, \bar{\sigma})$,

$$\begin{aligned} &\mathbb{E}_{\sigma_v}\left[\theta u_v - \zeta_t C(u_v) + \hat{\beta}_{t+1} k_{t+1}(w_v) - \chi_t W_t(\sigma_v)\right] \\ &\quad \ge \bar{u}_v - \zeta_t C(\bar{u}_v) + \hat{\beta}_{t+1} k_{t+1}(\bar{w}_v) - \chi_t W_t(\bar{\sigma}). \end{aligned} \tag{47}$$

First, suppose that utility is bounded below, without loss of generality by zero. From Lemma 7, $u_v(m) \to 0$ and $w_v(m) \to 0$ for any m as $v \to 0$, and therefore $\bar{u}_v$ and $\bar{w}_v$ also converge to zero. Hence in the limit equation (47) becomes

$$\limsup_{v \to \underline{v}} \chi_t \left(W_t(\bar{\sigma}) - W_t(\sigma_v)\right) \ge 0. \tag{48}$$

Since $W_t(\sigma_v) \ge W_t(\bar{\sigma})$ by Lemma 2 and $\chi_t > 0$ when (10) binds in periods t, $\lim_{v \to \underline{v}} W_t(\sigma_v)$ exists and satisfies $\lim_{v \to \underline{v}} W_t(\sigma_v) = W_t(\bar{\sigma})$. $W_t(\sigma)$ is continuous in σ and achieves its minimum only at uninformative reporting strategies by Lemma 2, therefore σ_v must converge to some uninformative strategy.

Now suppose that utility is unbounded below. By the mean value theorem

$$\begin{aligned} \zeta_t\left(C(u_v(m)) - C(\bar{u}_v)\right) &= \zeta_t C'(\tilde{u}_v(m))\left(u_v(m) - \bar{u}_v\right), \\ \hat{\beta}_{t+1}\left(k_{t+1}(w_v) - k_{t+1}(\bar{w}_v)\right) &= \hat{\beta}_{t+1} k'_{t+1}(\tilde{w}_v(m))\left(w_v(m) - \bar{w}_v\right), \end{aligned}$$

for some $\tilde{u}_v(m)$ that takes values between $u_v(m)$ and $\bar{u}_v$ and for some $\tilde{w}_v(m)$ that takes values between $w_v(m)$ and $\bar{w}_v$. By construction, $u_v(m_1) \le \bar{u}_v \le u_v(m_{N_v})$ and $w_v(m_1) \ge \bar{w}_v \ge w_v(m_{N_v})$, therefore, $u_v(m_1) \le \tilde{w}_v(m) \le u_v(m_{N_v})$ and $w_v(m_1) \ge \tilde{w}_v(m) \ge w_v(m_{N_v})$. From (28) and (29) $\lim_{v \to -\infty} C'(\tilde{u}_v(m)) = 0$ and $\lim_{v \to -\infty} k'_{t+1}(\tilde{w}_v(m)) = \beta/\hat{\beta}_{t+1}$ for all m. We have

$$\begin{aligned} &\lim_{v \to -\infty} \mathbb{E}_{\sigma_v}\left[\theta(u_v - \bar{u}_v) - \zeta_t\left(C(u_v) - C(\bar{u}_v)\right) + \hat{\beta}_{t+1}\left(k_{t+1}(w_v) - k_{t+1}(\bar{w}_v)\right)\right] \\ = &\lim_{v \to -\infty} \mathbb{E}_{\sigma_v}\left[-\zeta_t C'(\tilde{u}_v)(u_v - \bar{u}_v) + \left(\hat{\beta}_{t+1} k'_{t+1}(\tilde{w}_v) - \beta\right)(w_v - \bar{w}_v)\right] \\ = &\ 0, \end{aligned}$$

where the second line uses the mean value theorem and (46), and the last line uses the fact that $(u_v(m) - \bar{u}_v)$ and $(w_v(m) - \bar{w}_v)$ are bounded for low v by Lemma 7. This implies that (48) holds when utility is unbounded and that σ_v converges to an uninformative strategy. ■

We now can prove Theorem 1.

Proof of Theorem 1. Suppose $N_v > 1$ for some realization of z and consider (u_v, w_v, σ_v) that solve (39) for such z. Let θ'_v be the largest θ such that $m_1 \in M_v(\theta'_v)$. Let $\bar{u}^w$ be the optimal allocation when priors $\bar{\sigma}$ are uninformative. Then

$$\begin{aligned}
& W_t(\sigma_v) - W_t(\bar{\sigma}) \\
= & \max_{\{u(m)\}_{m \in M_v}} \sum_{m,\theta \in M_v \times \Theta} \left[(\theta u(m) - \lambda_t^w C(u(m))) - (\theta \bar{u}^w - \lambda_t^w C(\bar{u}^w)) \right] \sigma_v(m|\theta) \pi(\theta) \\
= & \max_{\{u(m)\}_{m \in M_v}} \left\{ \left[\mathbb{E}_{\sigma_v}[\theta|m_1] u(m_1) - \lambda_t^w C(u(m_1)) - (\mathbb{E}_{\sigma_v}[\theta|m_1] \bar{u}^w - \lambda_t^w C(\bar{u}^w)) \right] \left(\sum_{\theta \le \theta'_v} \sigma_v(m_1|\theta) \pi(\theta) \right) \right. \\
& \left. + \sum_{m \ge m_2} \left[\mathbb{E}_{\sigma_v}[\theta|m] u(m) - \lambda_t^w C(u(m)) - (\mathbb{E}_{\sigma_v}[\theta|m] \bar{u}^w - \lambda_t^w C(\bar{u}^w)) \right] \left(\sum_{\theta \ge \theta'_v} \sigma_v(m|\theta) \pi(\theta) \right) \right\}
\end{aligned}$$

All terms in square brackets are non-negative since the choice $u(m) = \bar{u}^w$ is feasible. They are strictly positive if $\mathbb{E}_{\sigma_v}[\theta|m] \neq 1$, since $\bar{u}^w$ is the optimal allocation for $\mathbb{E}[\theta|m] = 1$. From Lemma 8 the left hand side of this expression goes to zero as $v \to \underline{v}$, therefore the right hand side should also go to zero. This is possible either if (a) $\mathbb{E}_{\sigma_v}[\theta|m_1] \to 1$ (and hence $\theta'_v \to \theta_{|\Theta|}$) and $\left(\sum_{\theta \ge \theta'_v} \sigma_v(m|\theta) \pi(\theta)\right) \to 0$ for all $m > m_1$, or (b) $\left(\sum_{\theta \le \theta'_v} \sigma_v(m_1|\theta) \pi(\theta)\right) \to 0$, $\mathbb{E}_{\sigma_v}[\theta|m_2] \to 1$ and $\left(\sum_{\theta \ge \theta'_v} \sigma_v(m|\theta) \pi(\theta)\right) \to 0$ for all $m > m_2$ (and hence $\theta'_v \to \theta_1$ and $m_2 \in M_v(\theta_1)$). The other possibilities are ruled out since if $\mathbb{E}_{\sigma_v}[\theta|m] \to 1$ and $\left(\sum_{\theta \ge \theta'_v} \sigma_v(m|\theta) \pi(\theta)\right) \nrightarrow 0$ for some $m > m_2$, then for some $m' \le m_2$ we would have $\left(\sum_{\theta \ge \theta'_v} \sigma_v(m'|\theta) \pi(\theta)\right) \nrightarrow 0$ and $\mathbb{E}_{\sigma_v}[\theta|m'] \nrightarrow 1$.[16] Since it is impossible to have $\theta_{|\Theta|}$ and θ_1 to be indifferent between more than two distinct allocations in the optimum, for any $\varepsilon > 0$ there must be some v_t^- such that for all $v \le v_t^-$ the solution has $N_v = 2$ and either (a) $m_1 \in M_v(\theta_{|\Theta|})$, $\sigma_v(m_2|\theta_{|\Theta|}) \le \varepsilon$, $\mathbb{E}_{\sigma_v}[\theta|m_2] = \theta_{|\Theta|} > 1$ and $\mathbb{E}_{\sigma_v}[\theta|m_1]$ arbitrarily close to 1, or (b) $m_2 \in M_v(\theta_1)$, $\sigma_v(m_1|\theta_1) \le \varepsilon$, $\mathbb{E}_{\sigma_v}[\theta|m_1] = \theta_1 < 1$ and $\mathbb{E}_{\sigma_v}[\theta|m_2]$ arbitrarily close to 1. In case (a) (37) implies that $u_v^w(m_2) = C'^{-1}(\theta_{|\Theta|}/\lambda_t^w)$ and $u_v^w(m_1) \to C'^{-1}(1/\lambda_t^w)$, and in case (b) $u_v^w(m_1) = C'^{-1}(\theta_1/\lambda_t^w)$ and $u_v^w(m_2) \to C'^{-1}(1/\lambda_t^w)$.

We can now rule cases (a) and (b) for v low enough. Consider case (b), case (a) is ruled out analogously. In this case $\sigma_v(m_1|\theta_1) < 1$ for v low enough and the optimality condition

[16] Note that θ_1 can be indiffererent between at most two distict allocations and therefore $\sigma_v(m|\theta_1) = 0$ for all $m > m_2$.

(30) is

$$
\begin{aligned}
& \theta_1 \left(u_v \left(m_2\right) - u_v \left(m_1\right)\right) - \zeta_t \left(C \left(u_v \left(m_2\right)\right) - C \left(u_v \left(m_1\right)\right)\right) \qquad (49) \\
& + \hat{\beta}_{t+1} \left[k_{t+1} \left(w_v \left(m_2\right)\right) - k_{t+1} \left(w_v \left(m_1\right)\right)\right] \\
= \; & \left[\left(\theta_1 u_v^w \left(m_2\right) - \lambda_t^w C \left(u_v^w \left(m_2\right)\right)\right) - \left(\theta_1 u_v^w \left(m_1\right) - \lambda_t^w C \left(u_v^w \left(m_1\right)\right)\right)\right].
\end{aligned}
$$

Function $\theta_1 u - \lambda_t^w C(u)$ is strictly convex and achieves its maximum $\hat{u}$ at $\hat{u} = C'^{-1}\left(\theta_1/\lambda_t^w\right) = u_v^w \left(m_1\right)$. Since $u_v^w\left(m_2\right)$ is bounded away from $u_v^w\left(m_1\right)$, the right hand side of (49) is strictly negative and bounded away from 0.

When utility is bounded below, Lemma 7 established that all terms on the left hand side of (49) go to zero as $v \to \underline{v}$, yielding a contradiction. When utility is unbounded below, substitute the indifference condition $\theta_1 \left(u_v \left(m_2\right) - u_v \left(m_1\right)\right) = \beta \left(w_v \left(m_1\right) - w_v \left(m_2\right)\right)$ into (49) and apply the mean value theorem

$$
\zeta_t \left(C \left(u_v \left(m_2\right)\right) - C \left(u_v \left(m_1\right)\right)\right) = \zeta_t C' \left(\tilde{u}\right) \left(u_v \left(m_2\right) - u_v \left(m_1\right)\right)
$$

and

$$
\begin{aligned}
& \hat{\beta}_{t+1} \left(k_{t+1} \left(w_v \left(m_2\right)\right) - k_{t+1} \left(w_v \left(m_1\right)\right)\right) - \beta \left(w_v \left(m_2\right) - w_v \left(m_1\right)\right) \\
= \; & \left(\hat{\beta}_{t+1} k'_{t+1} \left(\tilde{w}\right) - \beta\right) \left(w_v \left(m_2\right) - w_v \left(m_1\right)\right)
\end{aligned}
$$

for some $\tilde{u} \in \left(u_v\left(m_1\right), u_v\left(m_2\right)\right)$ and $\tilde{w} \in \left(w_v\left(m_2\right), w_v\left(m_1\right)\right)$. The terms $\left(u_v\left(m_2\right) - u_v\left(m_1\right)\right)$ and $\left(w_v\left(m_2\right) - w_v\left(m_1\right)\right)$ are bounded for small v by Lemma 7 and $\lim_{v \to -\infty} \zeta_t C'\left(\tilde{u}\right) = 0$, $\lim_{v \to -\infty} \left(\hat{\beta}_{t+1} k'_{t+1}\left(\tilde{w}\right) - \beta\right) = 0$ by Lemma 6. Therefore, the left hand side of (49) converges to zero for utilities unbounded below, yielding a contradiction. This proves that $N_v = 1$ and σ_v is uninformative for all sufficiently low v independently of the realization of z. For uninformative σ_v, there is a unique $\left(u_v, w_v\right)$ that solves (39) by strict convexity of $u - \zeta_t C(u)$.

■

7.3.2 Proofs for high v

We prove Corollary 2, since Theorem 2 is a special case of it.

Corollary 2. Suppose v is sufficiently high so that $\gamma_v = k'_t(v) < 1$. We first rule out the case that θ_1 and θ_2 send the same message with probability 1 for high v. From (41) and (44) we have that there is $\vartheta_v \geq 0$ such that

$$
\frac{\hat{\beta}_{t+1}}{\beta} k'_{t+1} \left(w_v \left(m_1\right)\right) = \gamma_v - \vartheta_v,
$$

and

$$\zeta_t C'\left(u_v\left(m_1\right)\right) = \left(1-\gamma_v\right)\mathbb{E}_{\sigma_v}\left[\theta|m_1\right] + \vartheta_v \geq \left(1-\gamma_v\right)\tilde{\theta} + \vartheta_v,$$

where $\tilde{\theta} = \left(\pi_1\theta_1 + \pi_2\theta_2\right)/\left(\pi_1+\pi_2\right) > \theta_1$. The last inequality follows from the fact that types θ_1 and θ_2 play message m_1 with probability 1.

Define function f as

$$f\left(x\right) \equiv \theta_1\left(u_v\left(m_1\right)-x\right) - \zeta_t C\left(u_v\left(m_1\right)-x\right) + \hat{\beta}_{t+1}k_{t+1}\left(w_v\left(m_1\right)+\frac{\theta_1}{\beta}x\right).$$

This function is concave with

$$f'\left(0\right) = -\left(\theta_1 - \zeta_t C'\left(u_v\left(m_1\right)\right)\right) + \theta_1\left(\frac{\hat{\beta}_{t+1}}{\beta}k'_{t+1}\left(w_v\left(m_1\right)\right)\right) > \left(1-\gamma_v\right)\left(\tilde{\theta}-\theta_1\right).$$

Let $\hat{x}_v$ be a solution to $f'\left(\hat{x}_v\right) = \left(1-\gamma_v\right)\left(\tilde{\theta}-\theta_1\right)/2$ and let $\left(\hat{u}_v,\hat{w}_v\right) = \left(u_v\left(m_1\right)-\hat{x}_v, w_v\left(m_1\right)+\frac{\theta_1}{\beta}\hat{x}_v\right)$. Since f is concave, $\hat{x}_v > 0$ and $\hat{u}_v < u_v\left(m_1\right), \hat{w}_v > w_v\left(m_1\right)$.

Claim 1. Allocation $\left(\hat{u}_v,\hat{w}_v\right)$ satisfies bounds (28) and (29).

Let x_v^* be a solution to $f'\left(x_v^*\right) = 0$. By concavity, $0 < \hat{x}_v < x_v^*$, so we establish the claim by proving the stronger statement that $u_v\left(m_1\right) - x_v^*$ and $w_v\left(m_1\right) + \frac{\theta_1}{\beta}x_v^*$ satisfy bounds (28) and (29). By definition,

$$\theta_1\left[1 - \frac{\hat{\beta}_{t+1}}{\beta}k'_{t+1}\left(w_v\left(m_1\right)+\frac{\theta_1}{\beta}x_v^*\right)\right] = \zeta_t C'\left(u_v\left(m_1\right)-x_v^*\right)$$

and by $x_v^* > 0$

$$\begin{aligned}
\zeta_t C'\left(u_v\left(m_1\right)-x_v^*\right) &< \zeta_t C'\left(u_v\left(m_1\right)\right), \\
\frac{\hat{\beta}_{t+1}}{\beta}k'_{t+1}\left(w_v\left(m_1\right)+\frac{\theta_1}{\beta}x_v^*\right) &\leq \frac{\hat{\beta}_{t+1}}{\beta}k'_{t+1}\left(w_v\left(m_1\right)\right).
\end{aligned}$$

These conditions imply that

$$\begin{aligned}
\theta_1\left(1-\gamma_v\right) &\leq \zeta_t C'\left(u_v\left(m_1\right)-x_v^*\right) < \theta_{|\Theta|}\left(1-\gamma_v\right), \\
\theta_1\left(1-\gamma_v\right) &\leq \theta_1\left[1-\frac{\hat{\beta}_{t+1}}{\beta}k'_{t+1}\left(w_v\left(m_1\right)+\frac{\theta_1}{\beta}x_v^*\right)\right] < \left(1-\gamma_v\right)\theta_{|\Theta|},
\end{aligned}$$

establishing the bounds (28) and (29).

Claim 2. $\left(1-\gamma_v\right)\hat{x}_v \to \infty$ as $v \to \bar{v}$.

In Supplementary material we showed (Lemma 12) that Assumption 1 implies that there are numbers B_v and $\hat{B}_v$ such that

$$\begin{aligned}
C'\left(u_v\left(m_1\right)\right) - C'\left(\hat{u}_v\right) &\leq \frac{B_v}{\left[C'\left(u_v\left(m_1\right)\right)\right]^2}\left[C'\left(u_v\left(m_1\right)\right)\right]^2\hat{x}_v, \qquad (50) \\
k'_{t+1}\left(w_v\left(m_1\right)\right) - k'_{t+1}\left(\hat{w}_v\right) &\leq \frac{\hat{B}_v}{\left[1-k'_{t+1}\left(\hat{w}_v\right)\right]^2}\left[1-k'_{t+1}\left(\hat{w}_v\right)\right]^2\hat{x}_v,
\end{aligned}$$

and if $\hat{u}_v \to (1-\beta)\bar{v}, w_v(m_1) \to \bar{v}$ then $B_v / [C'(u_v(m_1))]^2 \to 0$ and $\hat{B}_v / \left[1 - k'_{t+1}(\hat{w}_v)\right]^2 \to 0$. The bounds from Claim 1 establish that $\hat{u}_v \to (1-\beta)\bar{v}$ as $v \to \bar{v}$. Since by (27) $k'_t(v) = \frac{\hat{\beta}_{t+1}}{\beta}\mathbb{E}_{\sigma_v} k'_{t+1}(w_v) \geq \frac{\hat{\beta}_{t+1}}{\beta} k'_{t+1}(w_v(m_1))$, this implies that $w_v(m_1) \to \bar{v}$ as $v \to \bar{v}$. Therefore by Lemma 12 the first terms on the right hand side of these expression go to zero as $v \to \bar{v}$.[17]

We have

$$\begin{aligned}
\frac{\tilde{\theta} - \theta_1}{2} &\leq \frac{1}{1-\gamma_v}\left[f'(0) - f'(\hat{x}_v)\right] \\
&= \frac{1}{1-\gamma_v}\left[\zeta_t\left\{C'(u_v(m_1)) - \zeta_t C'(\hat{u}_v)\right\} + \theta_1 \frac{\hat{\beta}_{t+1}}{\beta}\left\{k'_{t+1}(w_v(m_1)) - k'_{t+1}(\hat{w}_v)\right\}\right] \\
&\leq \left\{\frac{B_v}{[C'(u_v(m_1))]^2}\left[\frac{C'(u_v(m_1))}{1-\gamma_v}\right]^2 + \frac{\hat{B}_v}{\left[1-k'_{t+1}(\hat{w}_v)\right]^2}\left[\frac{1-k'_{t+1}(\hat{w}_v)}{1-\gamma_v}\right]^2\right\}(1-\gamma_v)\hat{x}_v.
\end{aligned}$$

From Claim 1, both $C'(u_v(m_1))/(1-\gamma_v)$ and $\left(1-k'_{t+1}(\hat{w}_v)\right)/(1-\gamma_v)$ are finite, therefore the expression in curly brackets goes to 0 as $v \to \bar{v}$. Therefore $(1-\gamma_v)\hat{x}_v \to \infty$.

Claim 3. $f(\hat{x}_v) - f(0) \to \infty$ as $v \to \bar{v}$.

Applying the mean value theorem,

$$f(\hat{x}_v) - f(0) = \frac{f'(\tilde{x}_v)}{1-\gamma_v}(1-\gamma_v)\hat{x}_v$$

for some $\tilde{x}_v \in (0, \hat{x}_v)$. Since f is concave, $f'(0) > f'(\tilde{x}_v) > f'(\hat{x}_v)$. Therefore $\frac{f(\tilde{x}_v)}{1-\gamma_v} \in \left[\frac{1}{2}\left(\tilde{\theta} - \theta_1\right), \left(\tilde{\theta} - \theta_1\right)\right]$. The result follows from Claim 2.

We are now ready to show that it is not optimal for types θ_1 and θ_2 to send the same message with probability 1 for high enough v. Suppose it is. Consider an alternative strategy $\hat{\sigma}_v$, where $\hat{\sigma}_v(\hat{m}|\theta_1) = 1$ for some message $\hat{m}$ that gives allocation $(\hat{u}_v, \hat{w}_v)$, and $\hat{\sigma}_v(m|\theta) = \sigma_v(m|\theta)$ for all $\theta \neq \theta_1, m$. Since $\hat{u} < u_v(m_1), \hat{w} > w_v(m_1)$ and $\theta_1 u_v(m_1) + \beta w_v(m_1) = \theta_1 \hat{u}_v + \beta \hat{w}_v$, this allocation is incentive compatible and delivers utility v to the agent. Then

$$\begin{aligned}
&\left\{\mathbb{E}_{\hat{\sigma}_v}\left[\theta u - \zeta_t C(u) + \hat{\beta} k_{t+1}(w)\right] - \chi_t W_t(\hat{\sigma}_v)\right\} - \left\{\mathbb{E}_{\sigma_v}\left[\theta u - \zeta_t C(u) + \hat{\beta} k_{t+1}(w)\right] - \chi_t W_t(\sigma_v)\right\} \\
&= \pi(\theta_1)\left\{f(\hat{x}_v) - f(0)\right\} + \chi_t\left\{W_t(\sigma_v) - W_t(\hat{\sigma}_v)\right\}.
\end{aligned}$$

The second term is finite since W_t is bounded by Lemma 2, therefore this expression is positive for high v from Claim 3. This contradicts the optimality of (u_v, w_v, σ_v).

Using these arguments we can also rule out type θ_1 sending the same message as θ_2 with any positive probability. Then the optimality condition (30) implies that

$$\left\{\theta_1 u_v(m_1) - \zeta_t C(u_v(m_1)) + \hat{\beta} k_{t+1}(w_v(m_1))\right\} - \left\{\theta_1 u_v(m_2) - \zeta_t C(u_v(m_2)) + \hat{\beta} k_{t+1}(w_v(m_2))\right\}$$

[17] When k_t is twice differentiable, this result can be established without using Lemma 12. In this case $k''_t(v) \leq -\zeta_t C''(u_v)$, and condition (31) implies that $\lim_{u \to (1-\beta)\bar{v}} C''(u)/[C'(u)]^2 = 0$ and $\lim_{v \to \bar{v}} k''_t(v)/[1-k'_t(v)]^2 = 0$. Claim 2 can then be established by appling the mean value theorem to the left hand side of (50).

is bounded because the expression on the right hand side of (30) is bounded for all σ. Therefore the value of this strategy can exceed the value of strategy when θ_1 and θ_2 send message m_2 with probability 1 by only a finite amount. But then the arguments of the previous paragraph lead to a contradiction.

Part 2 of Corollary 2 is proven using analogous arguments. Here we consider function

$$F(x) \equiv \theta_{|\Theta|}\left(u_v\left(m_{|M_v|}\right)+x\right)-\zeta_t C\left(u_v\left(m_{|M_v|}\right)+x\right)+\hat{\beta}_{t+1}k_{t+1}\left(w_v\left(m_{|M_v|}\right)-\frac{\theta_{|\Theta|-1}}{\beta}x\right).$$

When $\pi\left(\theta_{|\Theta|-1}\right)\left(\theta_{|\Theta|}-\theta_{|\Theta|-1}\right)>\left(\pi\left(\theta_{|\Theta|-1}\right)+\pi\left(\theta_{|\Theta|}\right)\right)\left(\theta_{|\Theta|-1}-\theta_{|\Theta|-2}\right)$ holds, we can show that $F'(0)=const\cdot(1-\gamma_v)$ and $\underline{\varrho}\geq 0$ ensures the boundary conditions. Previous arguments establish that $F(\hat{x}_v)-F(0)\to\infty$. Note that the perturbation we consider leaves the same allocation for type $\theta_{|\Theta|-1}$, and gives an allocation $\left(u_v\left(m_{|M_v|}\right)+\hat{x}_v, w_v\left(m_{|M_v|}\right)-\frac{\theta_{|\Theta|-1}}{\beta}\hat{x}_v\right)$ to type $\theta_{|\Theta|}$. This is incentive compatible but gives a higher value than v. The contradiction then follows from the fact that $k_t(v)$ is a decreasing function for high enough v. $\blacksquare$

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8 Supplementary material

8.1 Proofs for Section 3

Lemma 9 *(a) Any PBE* $(\boldsymbol{\sigma}, \boldsymbol{\sigma}_G, \mathbf{p})$ *is payoff-equivalent to a PBE* $(\boldsymbol{\sigma}', \boldsymbol{\sigma}'_G, \mathbf{p}')$ *where* $\boldsymbol{\sigma}'$ *satisfies*

$$\sigma'_t\left(\cdot\left|S^{t-1}, M_t, \check{h}^t, \left(\tilde{\theta}^{t-1}, \theta_t\right)\right.\right) = \sigma'_t\left(\cdot\left|S^{t-1}, M_t, \check{h}^t, \left(\hat{\theta}^{t-1}, \theta_t\right)\right.\right) \text{ for all } \tilde{\theta}^{t-1}, \hat{\theta}^{t-1}, \theta_t.$$

(b) Suppose $(\boldsymbol{\sigma}, \boldsymbol{\sigma}_G, \mathbf{p})$ *is a PBE and suppose that for some* $\left(S^{t-1}, M_t\right)$ *and personal histories* $\hat{h}^t, \tilde{h}^t \in H^t$

$$\mathbb{E}_{(\boldsymbol{\sigma}, \boldsymbol{\sigma}_G)}\left[\sum_{s=t+1}^{\infty} \beta^{s-t-1}\theta_s u_s \middle| S^{t-1}, M_t, \hat{h}^t\right] = \mathbb{E}_{(\boldsymbol{\sigma}, \boldsymbol{\sigma}_G)}\left[\sum_{s=t+1}^{\infty} \beta^{s-t-1}\theta_s u_s \middle| S^{t-1}, M_t, \tilde{h}^t\right]. \quad (51)$$

Then there exists a PBE $(\boldsymbol{\sigma}', \boldsymbol{\sigma}'_G, \mathbf{p}')$ *payoff equivalent to* $(\boldsymbol{\sigma}, \boldsymbol{\sigma}_G, \mathbf{p})$ *with a property that for all* $(z_{t+1}, m_{t+1}, ..., z_{t+s})$, *all* $S^{t+s} \supset \left(S^{t-1}, M_t\right)$, M_{t+s}

$$\begin{aligned}
\sigma'_{t+s}\left(\cdot|S^{t+s-1}, M_{t+s}, \hat{h}^t, z_{t+1}, m_{t+1}, ..., z_{t+s}\right) &= \sigma'_{t+s}\left(\cdot|S^{t+s-1}, M_{t+s}, \tilde{h}^t, z_{t+1}, m_{t+1}, ..., z_{t+s}\right), \\
u'_{t+s}\left(S^{t+s-1}, M_{t+s}, \hat{h}^t, z_{t+1}, m_{t+1}, ..., z_{t+s}, m_{t+s}\right) &= u'_{t+s}\left(S^{t+s-1}, M_{t+s}, \tilde{h}^t, z_{t+1}, m_{t+1}, ..., z_{t+s}, m_{t+s}\right).
\end{aligned}$$

Proof. (part a). Define $\check{\eta}_t : \mathcal{S}^{t-1} \times \mathcal{M} \to \Delta\left(\check{H}^t \times \Theta^t\right)$ analogously to the definition of $\eta_t : \mathcal{S}^t \to \Delta\left(H^t \times \Theta^t\right)$ in the text. Let B be a Borel set of $\check{H}^t$. For all B such that $\int_{\Theta^{t-1}} \check{\eta}_t\left(B, \left(d\theta^{t-1}, \theta_t\right)|S^{t-1}, M_t\right) > 0$ define strategy $\boldsymbol{\sigma}'$ by

$$\sigma'_t\left(\cdot\left|S^{t-1}, M_t, B, \left(\hat{\theta}^{t-1}, \theta_t\right)\right.\right) = \frac{\int_{\Theta^{t-1}} \sigma_t\left(\cdot\left|S^{t-1}, M_t, B, \left(\theta^{t-1}, \theta_t\right)\right.\right) \check{\eta}_t\left(B, \left(d\theta^{t-1}, \theta_t\right)|S^{t-1}, M_t\right)}{\int_{\Theta^{t-1}} \check{\eta}_t\left(B, \left(d\theta^{t-1}, \theta_t\right)|S^{t-1}, M_t\right)} \text{ all } \hat{\theta}^{t-1}.$$

For B such that $\int_{\Theta^{t-1}} \check{\eta}_t\left(B, \left(d\theta^{t-1}, \theta_t\right)|S^{t-1}, M_t\right) = 0$, set $\sigma'_t\left(\cdot\left|S^{t-1}, M_t, A, \left(\hat{\theta}^{t-1}, \theta_t\right)\right.\right) = \sigma'_t\left(\cdot\left|S^{t-1}, M_t, A, \left(\tilde{\theta}^{t-1}, \theta_t\right)\right.\right)$ for any $\tilde{\theta}^{t-1}$.

By construction, $\sigma'_t\left(\cdot\left|S^{t-1}, M_t, \check{h}^t, \left(\tilde{\theta}^{t-1}, \theta_t\right)\right.\right) = \sigma'_t\left(\cdot\left|S^{t-1}, M_t, \check{h}^t, \left(\hat{\theta}^{t-1}, \theta_t\right)\right.\right)$ for all $\tilde{\theta}^{t-1}, \hat{\theta}^{t-1}$. For any Borel set A of Θ^t, $\sigma'_t\left(A\left|\cdot, \left(\theta^{t-1}, \theta_t\right)\right.\right) > 0$ only if $\sigma_t\left(A\left|\cdot, \left(\hat{\theta}^{t-1}, \theta_t\right)\right.\right) > 0$ for some $\hat{\theta}^{t-1}$. Since any agent with a history $\left(\check{h}^t, \left(\tilde{\theta}^{t-1}, \theta_t\right)\right)$ can replicate the strategy of the agent with a history $\left(\check{h}^t, \left(\hat{\theta}^{t-1}, \theta_t\right)\right)$ and achieve the same payoff as that agent, and $\sigma_t\left(A\left|\cdot, \left(\hat{\theta}^{t-1}, \theta_t\right)\right.\right)$ is the optimal choice of agent $\left(\check{h}^t, \left(\hat{\theta}^{t-1}, \theta_t\right)\right)$, the new strategy $\boldsymbol{\sigma}'$ satisfies agents' best response constraint (4).

Strategies $\boldsymbol{\sigma}'$ induce distribution $\boldsymbol{\mu}'$ which satisfies $\mu'_t = \mu_t$ for all $\left(S^{t-1}, M_t\right)$, hence the government strategy $\boldsymbol{\sigma}_G$ satisfies feasibility (2) if agents play $\boldsymbol{\sigma}'$. Any posterior belief $\mathbf{p}'$ that satisfies (3) also satisfies $\int_{\Theta^{t-1}} p_t'\left(\left(\theta^{t-1},\theta\right)|\cdot\right) d\theta^{t-1} = \int_{\Theta^{t-1}} p_t\left(\left(\theta^{t-1},\theta\right)|\cdot\right) d\theta^{t-1}$. Government's payoff is

$$\begin{aligned}
\mathbb{E}_{(\boldsymbol{\sigma}',\boldsymbol{\sigma}_G)}\left[\theta_t u_t|S^t\right] &= \int_{H^t}\left[\int_{\Theta} p_t'\left(\theta|S^t,h^t\right)\theta d\theta\right] u_t\left(h^t\right)\mu_t'\left(dh^t|S^t\right) \\
&= \int_{H^t}\left[\int_{\Theta} p_t\left(\theta|S^t,h^t\right)\theta d\theta\right] u_t\left(h^t\right) d\mu_t\left(dh^t|S^t\right) = \mathbb{E}_{(\boldsymbol{\sigma},\boldsymbol{\sigma}_G)}\left[\theta_t u_t|S^t\right],
\end{aligned}$$

and therefore $(\boldsymbol{\sigma}',\boldsymbol{\sigma}_G',\mathbf{p}')$ achieves the same payoff as $(\boldsymbol{\sigma},\boldsymbol{\sigma}_G,\mathbf{p})$. No strategy $\boldsymbol{\sigma}_G'$ gives a higher payoff to the government that strategy $\boldsymbol{\sigma}_G$ when agents play $\boldsymbol{\sigma}'$ because otherwise $\boldsymbol{\sigma}_G'$ would give a higher payoff to the government that strategy $\boldsymbol{\sigma}_G$ when agents play $\boldsymbol{\sigma}$. Therefore $(\boldsymbol{\sigma}',\boldsymbol{\sigma}_G',\mathbf{p}')$ is a PBE that is payoff equivalent to $(\boldsymbol{\sigma},\boldsymbol{\sigma}_G,\mathbf{p})$.

(part b). For simplicity we drop explicit dependence on $\left(S^t, M_{t+1}\right)$ and assume that $\mu_t\left(\tilde{h}^t\right), \mu_t\left(\hat{h}^t\right) > 0$. Let $\alpha = \mu_t\left(\tilde{h}^t\right)/\left(\mu_t\left(\tilde{h}^t\right)+\mu_t\left(\hat{h}^t\right)\right)$ and define $\phi' : [0,\alpha] \to [0,1]$ by $\phi'(z) = z/\alpha$ and $\phi'' : (\alpha,1] \to [0,1]$ by $\phi''(z) = (z-\alpha)/(1-\alpha)$. Define strategies $(\boldsymbol{\sigma}',\boldsymbol{\sigma}_G')$ for all $s \geq 1$, $h^t \in \left\{\hat{h}^t,\tilde{h}^t\right\}$, θ^{t+s} as

$$\begin{aligned}
u_{t+s}'\left(h^t, z_{t+1}, m_{t+1}, ..., m_{t+s}\right) &= u_{t+s}^*\left(\tilde{h}^t, \phi'\left(z_{t+1}\right), m_{t+1}, ..., m_{t+s}\right), \\
\sigma_{t+s}'\left(\cdot|h^t, z_{t+1}, m_{t+1}, ..., z_{t+s}; \theta^{t+s}\right) &= \sigma_{t+s}^*\left(\cdot|\tilde{h}^t, \phi'\left(z_{t+1}\right), m_{t+1}, ..., z_{t+s}; \theta^{t+s}\right)
\end{aligned}$$

if $z_{t+1} \leq \alpha$ and

$$\begin{aligned}
u_{t+s}'\left(h^t, z_{t+1}, m_{t+1}, ..., m_{t+s}\right) &= u_{t+s}^*\left(\hat{h}^t, \phi''\left(z_{t+1}\right), m_{t+1}, ..., m_{t+s}\right), \\
\sigma_{t+s}'\left(\cdot|h^t, z_{t+1}, m_{t+1}, ..., z_{t+s}; \theta^{t+s}\right) &= \sigma_{t+s}^*\left(\cdot|\hat{h}^t, \phi''\left(z_{t+1}\right), m_{t+1}, ..., z_{t+s}; \theta^{t+s}\right)
\end{aligned}$$

if $z_{t+1} > \alpha$ and $u_s' = u_s, \sigma_s' = u_s$ for all other histories and periods s.

Agents with histories $\tilde{h}^t, \hat{h}^t$ could have replicated each other strategies after period t in PBE $(\boldsymbol{\sigma},\boldsymbol{\sigma}_G,\mathbf{p})$, so they must have been indifferent between them. Profile $\boldsymbol{\sigma}'$ gives them the same utility for all histories following $\left\{\hat{h}^t,\tilde{h}^t\right\}$ leaving all other histories unchanged, therefore it is incentive compatible, i.e. satisfies (4). Strategy profile $\boldsymbol{\sigma}'$ induces $\boldsymbol{\mu}'$. It assigns the same probability for any realization of u_t as $\boldsymbol{\mu}$, therefore feasibility constraint (2) is satisfied. For any $\mathbf{p}'$ consistent with Bayes' rule, $\mathbb{E}_{\boldsymbol{\sigma}}\left[\theta_t|h^t\right] = \mathbb{E}_{\boldsymbol{\sigma}'}\left[\theta_t|h^t\right]$ for all $h^t \in H^t$ hence (5) is satisfied. Therefore $(\boldsymbol{\sigma}',\boldsymbol{\sigma}_G',\mathbf{p}')$ is a PBE which is payoff equivalent to $(\boldsymbol{\sigma},\boldsymbol{\sigma}_G,\mathbf{p})$. ■

8.2 Arguments for maximization problem (17) and recursive formulation

We first discuss convexity assumptions in problem (17). We have $\sigma_t : \breve{H}^t \times \Theta \to \Delta(M_t)$, $u_t : H^t \to [(1-\beta)\underline{v}, (1-\beta)\bar{v}]$. Let Υ_t be a space of all such (σ_t, u_t), and let υ_t be a probability measure on Υ_t. For any υ_t and random variable $x : \Upsilon_t \times \Theta \to \mathbb{R}$ we can define $\mathbb{E}_{\upsilon_t} x = \int_{\Upsilon_t \times M_t \times \Theta} x\left(u_t\left(m\right), \sigma_t, \theta\right) \sigma_t\left(dm|\theta\right) d\pi d\upsilon$ and for an arbitrary σ'_t let $\mathbb{E}_{\upsilon_t \circ \sigma'_t} x = \int_{\Upsilon_t \times M_t \times \Theta} x\left(u_t\left(m\right), \sigma_t, \theta\right) \sigma'_t\left(dm|\theta\right) d\pi d\upsilon$. As before let $\boldsymbol{\upsilon}$ to denotes the infinite sequence $\{\upsilon_t\}_{t=0}^{\infty}$ and extend the definition of expectation $\mathbb{E}_{\boldsymbol{\upsilon} \circ \boldsymbol{\sigma}}$ to $V \times \Upsilon^\infty \times M^\infty \times \Theta^\infty$. With this notation, the modified problem (9) can be written as

$$\max_{\boldsymbol{\upsilon}} \mathbb{E}_{\boldsymbol{\upsilon}} \sum_{t=0}^{\infty} \beta^t \theta u$$

subject to

$$\mathbb{E}_{\boldsymbol{\upsilon}} C\left(u\right) \leq e \tag{52}$$

$$\mathbb{E}_{\boldsymbol{\upsilon}} \sum_{s=t}^{\infty} \beta^{s-t} \theta u \geq \mathbb{E}_{\boldsymbol{\upsilon}} W_t + \frac{\beta}{1-\beta} U\left(e\right) \text{ for all } t, \tag{53}$$

$$\mathbb{E}_{\boldsymbol{\upsilon}} \sum_{t=0}^{\infty} \beta^t \theta u \geq \mathbb{E}_{\boldsymbol{\upsilon} \circ \boldsymbol{\sigma}'} \sum_{t=0}^{\infty} \beta^t \theta u \text{ for all } \boldsymbol{\sigma}' \tag{54}$$

and

$$\mathbb{E}_{\boldsymbol{\upsilon}} \left[\sum_{t=0}^{\infty} \beta^t \theta u \middle| v \right] \geq v. \tag{55}$$

Note that we wrote the promise keeping constraint (55) as inequality. We do this more general version to prove some technical results on properties of the Lagrange multipliers. We still maintain the assumption that distribution ψ is such that in the optimum it holds with equality.

The objective function and constraints are linear in υ and therefore convex. Let $\left\{\beta^t \zeta_t^*\right\}_t$ and $\left\{\beta^t \chi_t^*\right\}_t$ be the Lagrange multipliers on (52) and (53). Then the constrained maximization problem can be written as

$$\max_{\boldsymbol{\upsilon}} \mathbb{E}_{\boldsymbol{\upsilon}} \sum_{t=0}^{\infty} \beta^t \left[\left(1 + \sum_{s=0}^{t} \chi_s^*\right) \theta u - \zeta_t^* C\left(u\right) \right] - \mathbb{E}_{\boldsymbol{\upsilon}} \sum_{t=0}^{\infty} \beta^t \chi_t^* W_t \tag{56}$$

subject to (54), (55). Redefining variables $\bar{\beta}_t = \beta^t \left(1 + \sum_{s=0}^{t} \chi_s^*\right)$, $\hat{\beta}_t = \bar{\beta}_t / \bar{\beta}_{t-1}$, $\zeta_t = \zeta_t^* / \left(1 + \sum_{s=0}^{t} \chi_s^*\right)$ and $\chi_t = \chi_t^* / \left(1 + \sum_{s=0}^{t} \chi_s^*\right) \geq 0$ we obtain (17) in a more general form. From the definitions of these variables, $\chi_t^* > 0$ implies $\hat{\beta}_t > \beta$ and $\chi_t > 0$. If $\limsup_{t \to \infty} \chi_t^* > 0$ then $\sum_{s=0}^{t} \chi_s^* \to \infty$ (Theorem 3.23 in Rudin (1976)) and $\bar{\beta}_t / \beta^t \to \infty$. The arguments of Sleet and Yeltekin (2008) establish that $\sum_{t=0}^{\infty} \bar{\beta}_t$, $\sum_{t=0}^{\infty} \beta^t \zeta_t^*$ and $\sum_{t=0}^{\infty} \beta^t \chi_t^*$ are all finite, and therefore $\sum_{t=0}^{\infty} \bar{\beta}_t \zeta_t$ and $\sum_{t=0}^{\infty} \beta_t \chi_t$ are also finite.

The Lagrange multiplier $\zeta_t > 0$ in any finite t. If it is not the case, then it is possible to give lifetime utility $\bar{v}$ to all families, which violates feasibility. The technical arguments simplify if, in addition, $\liminf \zeta_t > 0$. Sufficient conditions for this result are given in the following lemma

Lemma 10 *Let $\boldsymbol{\upsilon}^*$ be the best PBE. Suppose either that υ_t^* converges to an invariant distribution or that U is unbounded above and* $\liminf_{u\to\infty} \frac{C(u)}{C'(u)} > 0$. *Then* $\liminf \zeta_t > 0$.

Proof. We first observe that it is incentive compatible to increase utility allocation for all histories by $\delta > 0$ and that this increase satisfies (55). First, suppose that U is unbounded above. For $\delta > 0$ define υ_t^δ by $\upsilon_t^\delta(\sigma_t, u_t + \delta) = \upsilon_t^*(\sigma_t, u_t)$ for all (σ_t, u_t). Since υ_t^* is optimal and perturbation $(1-\alpha)\upsilon_t^\delta + \alpha\upsilon_t^*$ is feasible, this perturbation cannot increase the value of (56) evaluated at υ_t^*, i.e.

$$\left[\int \left[\left(1 + \sum_{s=0}^{t} \chi_s^*\right)\theta u - \zeta_t^* C(u)\right] \sigma(dm|\theta)\, d\pi\right] d\left[\upsilon_t^\delta - \upsilon_t^*\right] \leq 0$$

From the definition of υ_t^δ,

$$\left(1 + \sum_{s=0}^{t} \chi_s^*\right)\delta - \zeta_t^* \int \left[C(u+\delta) - C(u)\right] d\upsilon_t^* \leq 0.$$

Since it should be true for all δ, it implies that

$$\mathbb{E}_{\upsilon_t^*} C'(u) \geq \frac{1}{\zeta_t}.$$

Suppose $\liminf_{u\to\infty} \frac{C(u)}{C'(u)} > 0$, which implies that there is $\tilde{u}$ and $\kappa > 0$ such that $\frac{C(u)}{C'(u)} \geq \kappa$ for all $u \geq \tilde{u}$. Feasibility implies

$$\begin{aligned} e &\geq \mathbb{E}_{\upsilon_t^*} C(u) = \mathbb{E}_{\upsilon_t^*}\left[C(u)\,|u < \tilde{u}\right] P_{\upsilon_t^*}(u < \tilde{u}) + \mathbb{E}_{\upsilon_t^*}\left[\frac{C(u)}{C'(u)} C'(u)\,|u \geq \tilde{u}\right] P_{\upsilon_t^*}(u \geq \tilde{u}) \\ &\geq \mathbb{E}_{\upsilon_t^*}\left[\frac{C(u)}{C'(u)} C'(u)\,|u \geq \tilde{u}\right] P_{\upsilon_t^*}(u \geq \tilde{u}) \geq \kappa \mathbb{E}_{\upsilon_t^*}\left[C'(u)\,|u \geq \tilde{u}\right] P_{\upsilon_t^*}(u \geq \tilde{u}). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E}_{\upsilon_t^*} C'(u) &= \mathbb{E}_{\upsilon_t^*}\left[C'(u)\,|u < \tilde{u}\right] P_{\upsilon_t^*}(u < \tilde{u}) + \mathbb{E}_{\upsilon_t^*}\left[C'(u)\,|u \geq \tilde{u}\right] P_{\upsilon_t^*}(u \geq \tilde{u}) \\ &\leq \left(\max_{u\in[0,\tilde{u}]} C'(u)\right) P_{\upsilon_t^*}(u < \tilde{u}) + \frac{e}{\kappa} \\ &\leq \max_{u\in[0,\tilde{u}]} C'(u) + \frac{e}{\kappa} \end{aligned}$$

From strict convexity of C, $C'(\tilde{u}) = \max_{u\in[0,\tilde{u}]} C'(u)$, therefore,

$$\zeta_t \geq \left(C'(\tilde{u}) + \frac{e}{\kappa}\right)^{-1} > 0.$$

This proves the second part of the lemma. Analogous arguments establish our result for invariant distribution since in the invariant distribution $\mathbb{E}_{\upsilon^*} C'(u)$ must be finite. ■

If assumption $\limsup \chi_t > 0$ is not satisfied, economy converges to that of Atkeson and Lucas (1992), who showed that for a wide range of cost functions C (including those satisfying conditions of Lemma 10) all the mass of the lifetime utilities eventually gets concentrated arbitrarily close to the lower bound $\underline{v}$, the result known as the immiseration. Immiseration violates constraint (16) and therefore $\limsup \chi_t > 0$ in all specifications considered by Atkeson and Lucas (1992).

8.2.1 Sketch of proofs of Propositions 1 and 2

We now adapt the arguments of Farhi and Werning (2007) to write problem (17) recursively. When ζ_t is strictly positive with $\liminf \zeta_t > 0$, the series $\{\zeta_t\}_t$ are bounded away from zero uniformly in t. The arguments of Lemma A2 of Farhi and Werning (2007) extend with minimal modifications to problem (17) to show that it can be written in a recursive form

$$k_t(v) = \max_{\upsilon \in \Upsilon_t} \mathbb{E}_\upsilon \left[\theta u - \zeta_t C(u) + \hat{\beta}_{t+1} k_{t+1}(w) - \chi_t W_t \right]$$

subject to

$$v = \mathbb{E}_\sigma [\theta u + \beta w]$$

$$\mathbb{E}_\upsilon [\theta u + \beta w] \geq \mathbb{E}_{\upsilon \circ \sigma'} [\theta u + \beta w] \text{ for all } \sigma',$$

where $k_t(v)$ is convex. To achieve convexity of this problem, it is sufficient to randomize between only two points in Υ_t (the arguments are identical to the proof of Lemma 3 in Acemoglu, Golosov and Tsyvinski (2008)), and hence this problem can be written as

$$\begin{aligned} k_t(v) \quad = \quad & \max_{\substack{(u',w',\sigma'),\\ (u'',w'',\sigma''),\\ \bar{z}\in[0,1]}} \bar{z} \left\{ \int_{M_t \times \Theta} \left[\theta u' - \zeta_t C(u') + \hat{\beta}_{t+1} k_{t+1}(w') \right] \sigma'(dm|\theta)\, d\pi(\theta) - \chi_t W_t(\sigma') \right\} \\ & + (1-\bar{z}) \left\{ \int_{M_t \times \Theta} \left[\theta u'' - \zeta_t C(u'') + \hat{\beta}_{t+1} k_{t+1}(w'') \right] \sigma''(dm|\theta)\, d\pi(\theta) - \chi_t W_t(\sigma'') \right\} \end{aligned}$$

subject to

$$v = \bar{z} \left\{ \int_{M_t \times \Theta} [\theta u' + \beta w'] \, \sigma'(dm|\theta)\, d\pi(\theta) \right\} + (1-\bar{z}) \left\{ \int_{M_t \times \Theta} [\theta u'' + \beta w''] \, \sigma''(dm|\theta)\, d\pi(\theta) \right\}$$

and

$$\begin{aligned} \int_{M_t \times \Theta} [\theta u' + \beta w'] \, \sigma'(dm|\theta)\, d\pi(\theta) \quad &\geq \quad \int_{M_t \times \Theta} [\theta u' + \beta w'] \, \tilde{\sigma}(dm|\theta)\, d\pi(\theta) \text{ for all } \tilde{\sigma} \\ \int_{M_t \times \Theta} [\theta u'' + \beta w''] \, \sigma''(dm|\theta)\, d\pi(\theta) \quad &\geq \quad \int_{M_t \times \Theta} [\theta u'' + \beta w''] \, \tilde{\sigma}(dm|\theta)\, d\pi(\theta) \text{ for all } \tilde{\sigma} \end{aligned}$$

This is the Bellman equation (20). Then Proposition 1 and 2 can be established by direct adaptation of arguments in Farhi and Werning (2007). We show all the results for k_0, for all other k_t the arguments are identical. For brevity we drop explicit conditioning of all expectations on v.

Continuity

Since $k_0(v)$ is concave, it is continuous on $(\underline{v},\bar{v})$. To show that it is also continuous on the boundaries, define the value function

$$k_0^*(v)=\frac{1}{\bar{\beta}_0}\max_{\boldsymbol{\sigma},\mathbf{u}}\mathbb{E}_{\boldsymbol{\sigma}}\sum_{t=0}^{\infty}\bar{\beta}_t\left[\theta_t u_t-\zeta_t C(u_t)\right]+\max_{\boldsymbol{\sigma}}\left(-\sum_{t=0}^{\infty}\bar{\beta}_t\chi_t W_t\right)$$

subject to

$$v=\mathbb{E}_{\boldsymbol{\sigma}}\sum_{t=0}^{\infty}\beta^t\theta_t u_t.$$

We have $k_0^*(v)\geq k_0(v)$ and $k_0^*(v)$ is continuous. If utility is bounded below, then at $v=\underline{v}$, the solution to k_0^* sets $u_t=U(0)$ for all t and σ to minimize W_t. This allocation is incentive compatible, therefore, $k_0^*(\underline{v})=k_0(\underline{v})$. Then continuity of $k_0^*(v)$ at $\underline{v}$ implies continuity of $k_0(v)$ at $\underline{v}$. If utility is bounded above, then $C(u_t)\to+\infty$ as $v\to\bar{v}$. Therefore $\lim_{v\to\bar{v}}k_0^*(v)=-\infty$, which implies $\lim_{v\to\bar{v}}k_0(v)=-\infty$.

Differentiability

The proof is an application of the Benveniste and Scheinkman theorem (Benveniste and Scheinkman (1979)). First, suppose that utility is unbounded. Fix any interior v_0 and let $(\mathbf{u}^*,\boldsymbol{\sigma}^*)$ be the optimal allocation for that v. For any v, let $\tilde{u}_0=u_0^*+(v-v_0)$, $\tilde{u}_t=u_t^*$ for all $t>0$. Since utility is unbounded, this perturbation is feasible. Then $(\tilde{\mathbf{u}},\boldsymbol{\sigma}^*)$ satisfies (11) and (12) for v. Let

$$\begin{aligned}V(v) &= \frac{1}{\bar{\beta}_0}\mathbb{E}_{\boldsymbol{\sigma}^*}\sum_{t=0}^{\infty}\bar{\beta}_t\left[\theta_t\tilde{u}_t-\zeta_t C(\tilde{u}_t)-\chi_t W_t\right] \\ &= \mathbb{E}_{\boldsymbol{\sigma}^*}\left[\theta_0\left(u_0^*+(v-v_0)\right)-\zeta_0 C\left(u_0^*+(v-v_0)\right)-\chi_0 W_0\right] \\ &\quad +\frac{1}{\bar{\beta}_0}\mathbb{E}_{\boldsymbol{\sigma}^*}\sum_{t=1}^{\infty}\bar{\beta}_t\left[\theta_t u_t^*-\zeta_t C(u_t^*)-\chi_t W_t\right]\end{aligned} \tag{57}$$

We have $k_0(v)\geq V(v)$. Function V is differentiable with $V'(v_0)=1-\zeta_0\mathbb{E}_{\boldsymbol{\sigma}^*}C'(u_0^*)$. Since k_0 is concave, by Benveniste-Scheinkman theorem (see Theorem 4.10 in Stokey, Lucas and Prescott (1989)), $k_0'(v)$ exists and satisfies $k_0'(v)=1-\zeta_0\mathbb{E}_{\boldsymbol{\sigma}^*}C'(u_0^*)\leq 1$.

To find the values of $k_0'(v)$ in the limit as v approaches $\pm\infty$, define a function

$$\bar{K}(v)=\max_{\mathbf{u},\boldsymbol{\sigma}}\mathbb{E}_{\boldsymbol{\sigma}}\sum_{t=0}^{\infty}\beta^t\left[\theta_t u_t-\zeta_t C(u_t)\right]$$

subject to (12). It it easy to show that $\bar{K}(v)$ is concave and $\lim_{v\to-\infty}\bar{K}'(v)=1$ and $\lim_{v\to\infty}\bar{K}'(v)=-\infty$.

Let $\upsilon_t=\max_{\theta\in\Theta,c\geq 0}[\theta U(c)-\zeta_t c]$ and $\bar{\omega}_t=\chi_t\min_\sigma W_t(\sigma)$. Then

$$
\begin{aligned}
& k_0(v)-\frac{1}{\bar{\beta}_0}\sum_{t=0}^{\infty}\bar{\beta}_t\upsilon_t=\frac{1}{\bar{\beta}_0}\mathbb{E}_{\boldsymbol{\sigma}^*}\sum_{t=0}^{\infty}\bar{\beta}_t\left[\theta_t u_t^*-\zeta_t C\left(u_t^*\right)-\upsilon_t\right]-\mathbb{E}_{\boldsymbol{\sigma}^*}\sum_{t=0}^{\infty}\bar{\beta}_t\chi_t W_t \\
\leq\ & \mathbb{E}_{\boldsymbol{\sigma}^*}\sum_{t=0}^{\infty}\beta^t\left[\theta_t u_t^*-\zeta_t C\left(u_t^*\right)-\upsilon_t\right]-\sum_{t=0}^{\infty}\bar{\beta}_t\bar{\omega}_t \\
\leq\ & \bar{K}(v)-\sum_{t=0}^{\infty}\beta^t\upsilon_t-\sum_{t=0}^{\infty}\bar{\beta}_t\bar{\omega}_t,
\end{aligned}
$$

where the first inequality follows from the fact that the expression in square brackets is negative and $\bar{\beta}_t/\bar{\beta}_0\geq\beta^t$ and the second inequality follows from the fact that $\bar{K}(v)$ maximizes $\mathbb{E}_{\boldsymbol{\sigma}}\sum_{t=0}^{\infty}\beta^t\left[\theta_t u_t-\zeta_t C(u_t)\right]$ without incentive constraints.

Since $k_0(v)\leq\bar{K}(v)+const$ and $\bar{K}(v)$ is concave, $\lim_{v\to\infty}\bar{K}'(v)=-\infty$ implies $\lim_{v\to\infty}k_0'(v)=-\infty$. Since $k_0'(v)\leq 1$ and $\lim_{v\to-\infty}\bar{K}'(v)=1$, $\lim_{v\to-\infty}k_0'(v)=1$.

Now suppose that utility is bounded below but unbounded above. Without loss of generality, assume that $U(c)\geq 0$. Then for any v_0, the allocation $\left(\frac{v}{v_0}\mathbf{u}^*,\boldsymbol{\sigma}^*\right)$ is feasible (since $\frac{v}{v_0}u_t^*>0$ is feasible), incentive compatible and attains v. Let $V(v)=\frac{1}{\bar{\beta}_0}\mathbb{E}_{\boldsymbol{\sigma}^*}\sum_{t=0}^{\infty}\bar{\beta}_t\left[\theta_t\frac{v}{v_0}u_t^*-\zeta_t C\left(\frac{v}{v_0}u_t^*\right)-\chi_t W_t\right]$. $V(v)$ is concave, differentiable and satisfies $k_0(v_0)=V(v_0)$, therefore, $k_0(v)$ is differentiable at v_0 by the Benveniste-Scheinkman theorem. A symmetric argument works if utility is bounded above (without loss of generality by 0) but not below. If utility is bounded above and below, a function V can be constructed separately for $v\leq v_0$ and $v>v_0$. This shows that $k_0(v)$ is differentiable.

To establish the value of the derivatives in the limits, note that when utility is bounded, function $\bar{K}(v)$ still provides an upper bound to $k_0(v)$ and $\lim_{v\to\bar{v}}\bar{K}'(v)=-\infty$, which implies that $\lim_{v\to\bar{v}}k_0'(v)=-\infty$. When utility is unbounded below, we can define V as in (57) for $v\leq v_0$, which shows that $k_0'(v_0)\leq\lim_{v\uparrow v_0}V(v_0)=1-\zeta_0\mathbb{E}_{\sigma^*}C'(u_0^*)\leq 1$. Then the same arguments as for unbounded utility establish that $\lim_{v\to-\infty}k_0'(v)=1$.

It remains to show the value of $\lim_{v\to\underline{v}}k_0'(v)$ when utility is bounded below, without loss of generality by 0. Let

$$
\underline{K}(v)=\frac{1}{\bar{\beta}_0}\max_{u_t\in\mathbb{R}_+}\sum_{t=0}^{\infty}\bar{\beta}_t\left[\left(\sum_{\theta\in\Theta}\pi(\theta)\theta\right)u_t-\zeta_t C(u_t)\right]-\frac{1}{\bar{\beta}_0}\sum_{t=0}^{\infty}\bar{\beta}_t\bar{\omega}_t
$$

subject to (12). Let γ_v be a Lagrange multiplier on (12) for a given v. The first order condition for u_t is

$$
1-\zeta_t C'(u_t)\leq\frac{\beta^t}{\bar{\beta}_t/\bar{\beta}_0}\gamma_v, \tag{58}
$$

where the inequality sign is due to the non-negativity constraint on u_t. This expression implies that $\gamma_v \geq 1$. Therefore for the utilities bounded below, $\lim_{v \to \underline{v}} k_0'(v) \geq 1$. If $\limsup \chi_t > 0$, then $\frac{\beta^t}{\beta_t/\beta_0} \to 0$ and there is some T such that $1 - \frac{\beta^T}{\beta_T/\beta_0}\gamma_v > 0$. For such T the optimality condition (58) is satisfied only for $u_{v,T} > 0$. This is impossible since $\lim_{v\to\underline{v}} u_{v,t} = 0$ for all t, therefore if $\limsup \chi_t > 0$, then $\lim_{v\to\underline{v}} \gamma_v = \infty$. From the envelope theorem, $\underline{K}'(v) = \gamma_v$.

Since the solution to this problem is incentive compatible, $k_0(v) \geq \underline{K}(v)$ and $k_0(\underline{v}) = \underline{K}(\underline{v})$. Therefore $\lim_{v\to\underline{v}} k_0'(v) \geq \lim_{v\to\underline{v}}' \underline{K}'(v) \geq 1$ with $\lim_{v\to\underline{v}} k_0'(v) = \infty$ if $\limsup \chi_t > 0$.

The arguments for the proof of Proposition 2 mirror the proof of Theorem 2 in Farhi and Werning (2007).

8.3 Intermediate steps used in the proof of Theorem 2 and Corollary 2

We start with preliminary results.

Lemma 11 *Suppose that f is continuous on some interval $[a,b]$ and one of its Dini derivatives is bounded. Then f is Lipschitz continuous on $[a,b]$.*

Proof. Without loss of generality suppose that $D^+ f(t)$, defined as

$$D^+ f(t) \equiv \lim \sup_{h\to 0^+} \frac{f(t+h) - f(t)}{h},$$

is bounded by $\bar{D}$. Let $\Psi_1(t) = f(t) + \bar{D}t$. It is continuous since f is continuous and $D^+\Psi_1(t) = D^+ f(t) + \bar{D} \geq 0$. By Proposition 5.2 in Royden (1988) Ψ_1 is nondecreasing, and therefore $t'' > t'$ implies $f(t'') - f(t') \geq -\bar{D}(t'' - t')$. Applying the same arguments to $\Psi_2(t) = -f(t) + \bar{D}t$ and combining with the previous result, we establish $|f(t'') - f(t')| \leq \bar{D}|t'' - t'|$ for all $t'', t' \in [a,b]$.
■

Lemma 12 *If Assumption 1 is satisfied, then*

$$\lim_{u\to(1-\beta)\bar{v}} \frac{C''(u)}{[C'(u)]^2} = 0. \tag{59}$$

In particular, for any $\underline{v}(1-\beta) < a < b < \bar{v}(1-\beta)$ there exists a real number $B_{a,b}$ such that

$$\left|C'(\hat{u}) - C'(\tilde{u})\right| \leq B_{a,b}|\hat{u} - \tilde{u}| \text{ for all } \hat{u}, \tilde{u} \in [a,b]. \tag{60}$$

Moreover, for any $\varepsilon > 0$, there is $\bar{a}$ such that $B_{a,b}/(C'(b))^2 < \varepsilon$ for all $b > a \geq \bar{a}$.

For any $\underline{v} < a < b < \bar{v}$ such that $k_t'(a) < 1$, function k_t' is Lipschitz continuous on $[a,b]$ and there exist a real number $\hat{B}_{a,b}$ such that

$$\left|k_t'(\hat{v}) - k_t'(\tilde{v})\right| \leq \hat{B}_{a,b}|\hat{v} - \tilde{v}| \text{ for all } \hat{v}, \tilde{v} \in [a,b].$$

Moreover, for any $\varepsilon > 0$, there is $\bar{a}$ such that $\hat{B}_{a,b}/(1 - k_t'(b))^2 < \varepsilon$ for all $b > a \geq \bar{a}$.

Proof. By definition $C(U(c)) = c$ for all c. Differentiate twice

$$C'(U(c))\,U'(c) = 1$$

and

$$C''(U(c))\left[U'(c)\right]^2 + C'(U(c))\,U''(c) = 0. \tag{61}$$

Substitute the first expression into the second and regroup

$$\frac{C''(U(c))}{\left[C'(U(c))\right]^2} = -\frac{U''(c)}{U'(c)}.$$

If Assumption 1 is satisfied, we obtain (59). Since U'' is continuous, so is C'' from (61).

For any $\hat{u}, \tilde{u} \in [a, b]$ with $\tilde{u} < \hat{u}$,

$$C'(\hat{u}) - C'(\tilde{u}) = \int_{\tilde{u}}^{\hat{u}} C''(u)\,du \leq (\hat{u} - \tilde{u}) \max_{u \in [a,b]} C''(u),$$

where maximum is well defined since C'' is continuous. Let $\hat{u}_{a,b} = \arg\max_{u \in [a,b]} C''(u)$ and $B_{a,b} = C''(\hat{u}_{a,b})$. Since $C''(\hat{u}_{a,b}) / \left[C'(\hat{u}_{a,b})\right]^2 \geq C''(\hat{u}_{a,b}) / \left[C'(b)\right]^2$ and $\hat{u}_{a,b} \to (1-\beta)\bar{v}$ as $a \to (1-\beta)\bar{v}$, condition (59) establishes (60).

Since function k_t is concave and differentiable, k_t' is continuous on $[a, b]$ (Corollary 25.5.1 in Rockafellar (1972)). Let D^+ be the right upper Dini derivative of k_t', defined at each v_0 as

$$D^+ k_t'(v_0) \equiv \limsup_{v \to v_0^+} \frac{k_t'(v) - k_t'(v_0)}{v - v_0}.$$

Claim 1. $D^+ k_t'(v_0)$ satisfies

$$0 \geq D^+ k_t'(v_0) \geq V''(v_0),$$

where $V(v)$ is defined in (57).

Note that by construction V is twice differentiable with $V''(v_0) = -\zeta_t \mathbb{E}_{\sigma_{v_0}}\left[C''(u_{v_0})\right]$, $V(v) \leq k_t(v)$ for all v with equality for $v = v_0$ and $V'(v_0) = k_t'(v_0)$. Since k_t' is decreasing, $0 \geq D^+ k_t'(v_0)$ by definition. Suppose $D^+ k_t'(v_0) < V''(v_0)$. Then there exists $\hat{v} > v_0$, such that for all $v \in (v_0, \hat{v})$, $k_t'(v) < V'(v)$. If this is not the case, there must exist a sequence v_n, with $v_n \to v_0^+$, such that $k_t'(v_n) \geq V'(v_n)$ or

$$\frac{k_t'(v_n) - k_t'(v_0)}{v_n - v_0} \geq \frac{V'(v_n) - V'(v_0)}{v_n - v_0} \text{ for all } v_n.$$

Taking limits and invoking twice differentiability of V,

$$D^+ k_t'(v_0) \geq \limsup_{n \to \infty} \frac{k_t'(v_n) - k_t'(v_0)}{v_n - v_0} \geq V''(v_0),$$

which contradicts the assumption.

If $k_t'(v) < V'(v)$ for all $v \in (v_0, \hat{v})$, then

$$\int_{v_0}^{\hat{v}} k_t'(v)\,dv < \int_{v_0}^{\hat{v}} V'(v)\,dv,$$

where the integrals are well defined since k_t and V are concave and hence absolutely continuous by Proposition 5.17 in Royden (1988). Integrating and using the fact that $k_t(v_0) = V(v_0)$, we obtain $k_t(\hat{v}) < V(\hat{v})$, establishing the contradiction. Therefore $D^+k_t'(v_0) \geq V''(v_0)$.

Claim 2. k_t' is Lipschitz continuous on $[a,b]$.

It is sufficient to show that $V''(v_0) = -\zeta_t \mathbb{E}_{\sigma_{v_0}}[C''(u_{v_0})]$ is bounded on $[a,b]$ and apply Lemma 11. From (28),

$$\left(1 - k_t'(a)\right)\theta_1 \leq \zeta_t C'(u_{v_0}) \leq \left(1 - k_t'(b)\right)\theta_{|\Theta|} \text{ for all } v_0 \in [a,b]. \tag{62}$$

Since $k_t'(a) < 1$, this bounds u_{v_0}. C'' achieves a maximum at that set, say at a point $\hat{u}_{a,b}$, which implies that $V''(v_0)$ is bounded by $\hat{B}_{a,b} = \zeta_t C''(\hat{u}_{a,b})$.

Claim 3. Lipschitz bound $\hat{B}_{a,b}$ satisfies the condition that for any $\varepsilon > 0$, there is $\bar{a}$ such that $\hat{B}_{a,b}/(1 - k_t'(b))^2 < \varepsilon$ for all $b > a \geq \bar{a}$.

As $a \to \bar{v}$, $k_t'(a) \to -\infty$ and therefore equation (62) implies that $\hat{u}_{a,b}$ gets arbitrarily close to $(1-\beta)\bar{v}$ for all a sufficiently high. By the first part of the lemma, this implies that $C''(\hat{u}_{a,b}) / [C'(\hat{u}_{a,b})]^2$ approaches zero for high a. Hence

$$\frac{\hat{B}_{a,b}}{[1 - k_t'(b)]^2} = \frac{\zeta_t C''(\hat{u}_{a,b})}{[C'(\hat{u}_{a,b})]^2} \left(\frac{C'(\hat{u}_{a,b})}{1 - k_t'(b)}\right)^2 \leq \frac{\zeta_t C''(\hat{u}_{a,b})}{[C'(\hat{u}_{a,b})]^2} \left(\frac{\theta_{|\Theta|}}{\zeta_t}\right)^2$$

also approaches 0 as $a \to \bar{v}$. ■

8.4 Proof of Proposition 4

Suppose C satisfies

$$C(u) = \frac{1}{a} u^a \text{ for } a > 1. \tag{63}$$

For all $x > 0$, define a function

$$k_t(v, x) = \frac{1}{\bar{\beta}_t} \max_{\mathbf{u}, \boldsymbol{\sigma}} \mathbb{E}_{\boldsymbol{\sigma}} \left[\sum_{s=0}^{\infty} \bar{\beta}_{s+t} \left(\theta_s u_s x^{a-1} - \zeta_{s+t} C(u_s) - x^a \chi_{s+t} W_s\right)\right]$$

subject to (11) and (12). For $x = 0$ we set

$$k_t(v, 0) = \frac{1}{\bar{\beta}_t} \max_{\mathbf{u}, \boldsymbol{\sigma}} \mathbb{E}_{\boldsymbol{\sigma}} \left[\sum_{s=0}^{\infty} \bar{\beta}_{s+t} \left(-\zeta_{s+t} C(u_s)\right)\right].$$

We prove several preliminary result first.

Lemma 13 *Suppose C satisfies (63). Then $k_t(v,x)$ is continuous in (v,x).*

Proof. For interior (v,x) it is immediate, so we show our result for boundary: if $(v_n,x_n)\to(v,0)$ then $k_t(v_n,x_n)\to k_t(v,0)$. Since $|k_t(v_n,x_n)-k_t(v,0)|\le|k_t(v_n,x_n)-k_t(v,x_n)|+|k_t(v,x_n)-k_t(v,0)|$, and $k_t(v,x)$ is continuous in v for all $x\ge 0$ by standard arguments, it is sufficient to establish that $k_t(v,x_n)\to k_t(v,0)$ as $x_n\to 0$.

We show our result for $k_0(v,x)$, the arguments are analogous for other periods. Let

$$\bar{K}(v,x)=\max_{\mathbf{u},\boldsymbol{\sigma}}\mathbb{E}_{\boldsymbol{\sigma}}\sum_{t=0}^{\infty}\beta^t\left[x^{a-1}\theta_t u_t-\zeta_t C(u_t)\right]$$

subject to (12). Analogously with the proof of Proposition 1, $\bar{K}(v,x)$ is finite for all $x\ge 0$ and $k_0(v,x)\le\bar{K}(v,x)+x^a\cdot const$, therefore $k_0(v,x)$ is bounded from above, and that bound can be chosen to be uniform for all x in the neighborhood of $x=0$. Function $k_0(v,x)$ is bounded below because $u\ge 0$, $C(u)\ge 0$ and W_t is bounded. Moreover, that bound can be chosen to be uniform for all x in the neighborhood of $x=0$.

Let $(\mathbf{u}^x,\boldsymbol{\sigma}^x)$ be a solution to $k_0(v,x)$ for a given x. We show next that $\sum_{t=0}^{\infty}\bar{\beta}_t u_t^x$ is bounded for all x in the neighborhood of $x=0$. Since $\zeta_t C(u)$ is convex, there are reals b_t' and $b_t''>0$ such that $-\zeta_t C(u)\le b_t'-\theta_{|\Theta|}b_t''u$ for all u. Since ζ_t is bounded away from zero, we can pick b' and b'' to be independent of t. Then

$$\begin{aligned}
\bar{\beta}_0 k_0(v,x) &= \mathbb{E}_{\boldsymbol{\sigma}^x}\sum_{t=0}^{\infty}\bar{\beta}_t\left(x^{a-1}\theta u_t^x-\zeta_t C(u_t^x)\right)-x^a\mathbb{E}_{\boldsymbol{\sigma}^x}\sum_{t=0}^{\infty}\bar{\beta}_t\chi_t W_t\\
&\le b'\sum_{t=0}^{\infty}\bar{\beta}_t+\mathbb{E}_{\boldsymbol{\sigma}^x}\sum_{t=0}^{\infty}\bar{\beta}_t\left(x^{a-1}\theta u_t^x-\theta_{|\Theta|}b''u_t^x\right)-x^a\mathbb{E}_{\boldsymbol{\sigma}^x}\sum_{t=0}^{\infty}\bar{\beta}_t\chi_t W_t\\
&\le b'\sum_{t=0}^{\infty}\bar{\beta}_t+\mathbb{E}_{\boldsymbol{\sigma}^x}\sum_{t=0}^{\infty}\bar{\beta}_t\left(x^{a-1}-b''\right)\theta u_t^x-x^a\mathbb{E}_{\boldsymbol{\sigma}^x}\sum_{t=0}^{\infty}\bar{\beta}_t\chi_t W_t.
\end{aligned}$$

For $x^{a-1}<b''$ this yields

$$0\le\mathbb{E}_{\boldsymbol{\sigma}^x}\sum_{t=0}^{\infty}\bar{\beta}_t\theta u_t^x\le\frac{b'}{b''-x^{a-1}}\sum_{t=0}^{\infty}\bar{\beta}_t-x^a\mathbb{E}_{\boldsymbol{\sigma}^x}\sum_{t=0}^{\infty}\bar{\beta}_t\chi_t W_t-\bar{\beta}_0 k_0(v,x).$$

Since $(\mathbf{u}^x,\boldsymbol{\sigma}^x)$ are optimal for x, incentive compatible and provide utility v to agent,

$$\mathbb{E}_{\boldsymbol{\sigma}^0}\left[\sum_{t=0}^{\infty}\bar{\beta}_t\left(-\zeta_t C\left(u_t^0\right)\right)\right]\ge\mathbb{E}_{\boldsymbol{\sigma}^x}\left[\sum_{t=0}^{\infty}\bar{\beta}_t\left(-\zeta_t C(u_t^x)\right)\right],$$

where the right hand side expression is well defined since $k_0(v,x)$ and $\mathbb{E}_{\boldsymbol{\sigma}^x}\sum_{t=0}^{\infty}\bar{\beta}_t\theta u_t^x$ are finite, which implies that $k_0(v,0)\ge\limsup_{x\to 0}\mathbb{E}_{\boldsymbol{\sigma}^x}\left[\sum_{t=0}^{\infty}\bar{\beta}_t\left(-\zeta_t C(u_t^x)\right)\right]$. At the same time

$$k(v,x)=\mathbb{E}_{\boldsymbol{\sigma}^x}\left[\sum_{t=0}^{\infty}\bar{\beta}_t\left(\theta_t u_t^x x^{a-1}-\zeta_t C(u_t^x)-x^a\chi_t W_t\right)\right]\ge\mathbb{E}_{\boldsymbol{\sigma}^0}\left[\sum_{t=0}^{\infty}\bar{\beta}_t\left(\theta_t u_t^0 x^{a-1}-\zeta_t C\left(u_t^0\right)-x^a\chi_t W_t\right)\right]$$

which implies that

$$\liminf_{x\to 0} k_0(v,x) = \liminf_{x\to 0} \mathbb{E}_{\boldsymbol{\sigma}^x}\left[\sum_{t=0}^{\infty}\bar{\beta}_t\left(-\zeta_t C\left(u_t^x\right)\right)\right] \geq k(v,0),$$

where again we used boundedness of $\mathbb{E}_{\boldsymbol{\sigma}^x}\sum_{t=0}^{\infty}\bar{\beta}_t\theta u_t^x$. Therefore $\lim_{x\to 0} k_0(v,x) = k_0(v,0)$ for all v. ■

First, we analyze the limiting case of $x=0$. It has a recursive structure

$$k_t(v,0) = \max_{u,w,\sigma}\mathbb{E}_{\sigma}\left[-\zeta_t C(u,0) + \hat{\beta}_{t+1}k_{t+1}(w,0)\right]$$

subject to (22) and (21). Any conditioning on z is redundant since $C(\cdot,u)$ is strictly convex and then the Revelation principle implies strict convexity of $k_t(\cdot,0)$. One can also easily show that $k_t(\cdot,0)$ is differentiable and decreasing.

Let $\left\{u_v^0(m), w_v^0(m), \sigma_v^0(m|\theta)\right\}_{\theta,m}$ be a solution to this problem.

Lemma 14 *Suppose that* $\left(u_v^0(m'), w_v^0(m')\right) \neq \left(u_v^0(m''), w_v^0(m'')\right)$ *for some* m', m'', $\sigma_v^0(m'|\theta) > 0$ *and*

$$\theta u_v^0(m') + \beta w_v^0(m') = \theta u_v^0(m'') + \beta w_v^0(m'').$$

Then

$$-\zeta_t C\left(u_v^0(m')\right) + \hat{\beta}_{t+1}k_{t+1}\left(w_v^0(m')\right) > -\zeta_t C\left(u_v^0(m'')\right) + \hat{\beta}_{t+1}k_{t+1}\left(w_v^0(m'')\right) \tag{64}$$

and $\sigma_v^0(m''|\theta) = 0$.

Proof. Suppose

$$-\zeta_t C\left(u_v^0(m')\right) + \hat{\beta}_{t+1}k_{t+1}\left(w_v^0(m')\right) < -\zeta_t C\left(u_v^0(m'')\right) + \hat{\beta}_{t+1}k_{t+1}\left(w_v^0(m'')\right).$$

Then setting $\tilde{\sigma}_v^0(m''|\theta) = \sigma_v^0(m''|\theta) + \sigma_v^0(m'|\theta)$, $\tilde{\sigma}_v^0(m'|\theta) = 0$ and leaving all other reporting strategies unchanged satisfies (22) and (21) and delivers a strictly higher value of $\mathbb{E}_{\sigma}\left[-\zeta_t C(u) + \hat{\beta}_{t+1}k_{t+1}(w,0)\right]$, contradicting optimality of $\sigma_v^0(m'|\theta) > 0$.

Suppose (64) holds with equality. Defined as $\tilde{u}_\alpha = \alpha u_v^0(m') + (1-\alpha)u_v^0(m'')$ and $\tilde{w}_\alpha = \alpha w_v^0(m') + (1-\alpha)w_v^0(m'')$ for some $\alpha \in (0,1)$. The new allocation satisfies (21) since θ is indifferent between $\left(u_v^0(m'), w_v^0(m')\right)$ and $\left(u_v^0(m''), w_v^0(m'')\right)$. It also satisfies (22) since for any $\hat{\theta} \neq \theta$ and $\hat{m}$ that $\hat{\theta}$ sends with a positive probability,

$$\hat{\theta}u_v^0(\hat{m}) + \beta w_v^0(\hat{m}) \geq \hat{\theta}u_v^0(m') + \beta w_v^0(m')$$

and

$$\hat{\theta}u_v^0(\hat{m}) + \beta w_v^0(\hat{m}) \geq \hat{\theta}u_v^0(m'') + \beta w_v^0(m'')$$

and therefore

$$\hat{\theta}u_v^0(\hat{m}) + \beta w_v^0(\hat{m}) \geq \hat{\theta}\tilde{u}_\alpha + \beta\tilde{w}_\alpha.$$

For any α, by strict concavity of $-C$ and k_{t+1}

$$\begin{aligned} -\zeta_t C(u_\alpha) + \hat{\beta}_{t+1}k_{t+1}(w_\alpha) &> -\zeta_t C\left(u_v^0(m')\right) + \hat{\beta}_{t+1}k_{t+1}\left(w_v^0(m')\right) \\ &= -\zeta_t C\left(u_v^0(m'')\right) + \hat{\beta}_{t+1}k_{t+1}\left(w_v^0(m'')\right). \end{aligned} \tag{65}$$

Augment the message space M_v with a message $m_\varnothing$. Define $\tilde{\sigma}(m_\varnothing|\theta) = \sigma_v^0(m'|\theta) + \sigma_v^0(m''|\theta)$ and $\tilde{\sigma}\left(m_\varnothing|\hat{\theta}\right) = 0$ for all $\hat{\theta} \neq \theta$, and $\tilde{\sigma}(m|\theta) = \sigma_v^0(m|\theta)$ for $m \notin \{m', m'', m_\varnothing\}$, $\tilde{\sigma}\left(m|\hat{\theta}\right) = \sigma_v^0\left(m|\hat{\theta}\right)$ for all $\hat{\theta} \neq \theta$ and all $m \neq m_\varnothing$. Similarly let $(\tilde{u}(m_\varnothing), \tilde{w}(m_\varnothing)) = (u_\alpha, w_\alpha)$ for any $\alpha \in (0,1)$ and $(\tilde{u}(m), \tilde{w}(m)) = \left(u_v^0(m), w_v^0(m)\right)$ for all $m \neq m_\varnothing$. That is we consider an augmented state space and a strategy in which type θ reports $m_\varnothing$ and receives (u_α, w_α) in all states in which she reported m', m'' leaving all other strategies and allocations unchanged. The 3-tuple $(\tilde{u}, \tilde{w}, \tilde{\sigma})$ is incentive compatible and delivers the same payoff v to the agent, but by (65) and the fact that $\tilde{\sigma}(m_\varnothing|\theta) > 0$ delivers strictly higher value to the planner. Therefore $\left(u_v^0, w_v^0, \sigma_v^0\right)$ cannot be optimal, leading to a contradiction. ■

This lemma shows that each type can receive only one distinct allocation. Under some additional conditions requiring types to be sufficiently spread out, we can also show that each θ receives a distinct allocation from other types.

Lemma 15 *Suppose condition (33) is satisfied. Then if $\sigma_v^0(m_i|\theta) > 0$ for some θ, then $\sigma_v^0(m_i|\theta') = 0$ for all $\theta' \neq \theta$.*

Proof. Previous lemma established that types can receive at most $|\Theta|$ distinct allocations, so we restrict attention to only $|\Theta|$ messages. Without loss of generality $u_v^0(m_1) \leq ... \leq u_v^0\left(m_{|\Theta|}\right)$. Suppose that there is an allocation $\left(u_v^0(m_n), w_v^0(m_n)\right)$ that two types receive with a positive probability. Due to the previous lemma, they must receive it with probability 1. Pick the highest type that receives an allocation which is also received by some lower type. To simplify notation, call that type θ_n and the single crossing property implies that θ_{n-1} also receives $\left(u_v^0(m_{n-1}), w_v^0(m_{n-1})\right) = \left(u_v^0(m_n), w_v^0(m_n)\right)$. For now assume that $u_v^0(m_{n-2}) < u_v^0(m_{n-1})$.

First, observe that it must be true that

$$\theta_{n+1}u_v^0(m_{n+1}) + \beta w_v^0(m_{n+1}) > \theta_{n+1}u_v^0(m_n) + \beta w_v^0(m_n).$$

Otherwise, if this inequality is weak, the fact that $\left(u_v^0\left(m_{n+1}\right), w_v^0\left(m_{n+1}\right)\right) \neq \left(u_v^0\left(m_n\right), w_v^0\left(m_n\right)\right)$ implies

$$\theta_n u_v^0\left(m_n\right) + \beta w_v^0\left(m_n\right) > \theta_n u_v^0\left(m_{n+1}\right) + \beta w_v^0\left(m_{n+1}\right).$$

But then $w_v^0\left(m_{n+1}\right)$ can be decreased and $w_v^0\left(m_n\right)$ increased while keeping $\pi\left(\theta_n\right) w_v^0\left(m_n\right) + \pi\left(\theta_{n+1}\right) w_v^0\left(m_{n+1}\right)$ constant. For small changes that will be incentive compatible, and strict concavity of $k_{t+1}(\cdot, 0)$ will imply that the perturbed allocation gives a higher value, contradicting optimality.

Choose $\varepsilon > 0$ small enough so that

$$\theta_{n+1} u_v^0\left(m_{n+1}\right) + \beta w_v^0\left(m_{n+1}\right) > \theta_{n+1} u_v^0\left(m_n\right) + \beta w_v^0\left(m_n\right) + \theta_{n+1} \frac{\pi\left(\theta_n\right)}{\pi\left(\theta_{n-1}\right)} \varepsilon.$$

Let $\delta_2(\varepsilon) = \frac{\theta_{n-1}}{\beta}\varepsilon$, $\delta_3(\varepsilon) = \frac{\theta_{n-1}-\theta_{n-2}}{\beta}\varepsilon$ and

$$\delta_1(\varepsilon) = \frac{1}{\beta}\pi\left(\theta_{n-1}\right)\left[\theta_n - \theta_{n-1}\right]\varepsilon + \frac{\theta_{n-1}-\theta_{n-2}}{\beta}\sum_{i=1}^{n-2}\pi\left(\theta_i\right)\varepsilon.$$

By construction, all δ are positive and $O(\varepsilon)$, and

$$\delta_1(\varepsilon) - \delta_3(\varepsilon) = \frac{1}{\beta}\left\{\pi\left(\theta_{n-1}\right)\left[\theta_n - \theta_{n-1}\right] - \left(\theta_{n-1} - \theta_{n-2}\right)\sum_{i=n-1}^{|\Theta|}\pi\left(\theta_i\right)\right\}\varepsilon \geq 0$$

if condition (33) is satisfied.

Consider an allocation $(\tilde{u}, \tilde{w})$ defined as

$$\begin{aligned}
\tilde{u}\left(m_n\right) &= u_v^0\left(m_n\right) + \frac{\pi\left(\theta_{n-1}\right)}{\pi\left(\theta_n\right)}\varepsilon, \\
\tilde{u}\left(m_{n-1}\right) &= u_v^0\left(m_{n-1}\right) - \varepsilon, \\
\tilde{u}\left(m_i\right) &= u_v^0\left(m_i\right) \text{ for } i \notin \{n-1, n\},
\end{aligned}$$

and

$$\begin{aligned}
\tilde{w}\left(\theta_i\right) &= w_v^0\left(\theta_i\right) + \delta_3(\varepsilon) - \delta_1(\varepsilon) \text{ for } i \leq n-2, \\
\tilde{w}\left(\theta_i\right) &= w_v^0\left(\theta_i\right) - \delta_1(\varepsilon) \text{ for } i > n, \\
\tilde{w}\left(\theta_n\right) &= w_v^0\left(\theta_n\right) - \frac{\pi\left(\theta_{n-1}\right)}{\pi\left(\theta_n\right)}\delta_2(\varepsilon) - \delta_1(\varepsilon), \\
\tilde{w}\left(\theta_{n-1}\right) &= w_v^0\left(\theta_{n-1}\right) + \delta_2(\varepsilon) - \delta_1(\varepsilon).
\end{aligned}$$

First, observe that $(\tilde{u}, \tilde{w})$ satisfy (21) since

$$\begin{aligned}
& \sum_{i=1}^{|\Theta|} \pi(\theta_i) \left[\theta_i \tilde{u}(m_i) + \beta \tilde{w}(m_i)\right] - \sum_{i=1}^{|\Theta|} \pi(\theta_i) \left[\theta_i u_v^0(m_i) + \beta w_v^0(m_i)\right] \\
= \; & \pi(\theta_{n-1})(\theta_n - \theta_{n-1})\varepsilon + \beta\delta_3(\varepsilon) \sum_{i=1}^{n-2} \pi(\theta_i) - \beta\delta_1(\varepsilon) \\
= \; & \pi(\theta_{n-1})(\theta_n - \theta_{n-1})\varepsilon + (\theta_{n-1} - \theta_{n-2})\varepsilon \sum_{i=1}^{n-2} \pi(\theta_i) - \left(\pi(\theta_{n-1})\left[\theta_n - \theta_{n-1}\right]\varepsilon + (\theta_{n-1} - \theta_{n-2}) \sum_{i=1}^{n-2} \pi(\theta_i)\varepsilon\right) \\
= \; & 0.
\end{aligned}$$

It also satisfies incentive compatibility. Note that for small ε we have $\tilde{u}(m_1) \leq ... \leq \tilde{u}\left(m_{|\Theta|}\right)$ and it suffices to check local downward incentive compatibility. We have

$$\begin{aligned}
& \theta_{n-1}\tilde{u}(m_{n-1}) + \beta\tilde{w}(m_{n-1}) - \theta_{n-1}\tilde{u}(m_n) - \beta\tilde{w}(m_n) \\
= \; & -\theta_{n-1}\left(1 + \frac{\pi(\theta_{n-1})}{\pi(\theta_n)}\right)\varepsilon + \left(1 + \frac{\pi(\theta_{n-1})}{\pi(\theta_n)}\right)\beta\delta_2(\varepsilon) = 0,
\end{aligned}$$

so the incentive constraint for type θ_{n-1} is satisfied. Also

$$\begin{aligned}
& \theta_{n-2}\tilde{u}(m_{n-2}) + \beta\tilde{w}(m_{n-2}) - \theta_{n-2}\tilde{u}(m_{n-1}) - \beta\tilde{w}(m_{n-1}) \\
= \; & \theta_{n-2}\varepsilon + (\theta_{n-1} - \theta_{n-2})\varepsilon - \theta_{n-1}\varepsilon = 0,
\end{aligned}$$

so the incentive for type θ_{n-2} is satisfied. Similar arguments hold for all the other incentive constraints. Finally

$$\begin{aligned}
& \sum_{i=0}^{|\Theta|} \pi(\theta_i) \left[-\zeta_t C(\tilde{u}(m_i)) + \hat{\beta}_{t+1} k_{t+1}(\tilde{w}(m_i))\right] - \sum_{i=0}^{|\Theta|} \pi(\theta_i) \left[-\zeta_t C\left(u_v^0(m_i)\right) + \hat{\beta}_{t+1} k_{t+1}\left(w_v^0(m_i)\right)\right] \\
= \; & \sum_{i=0}^{n-2} \pi(\theta_i) k'_{t+1}(\tilde{w}(m_i))(\delta_3(\varepsilon) - \delta_1(\varepsilon)) + \sum_{i=n+1}^{|\Theta|} \pi(\theta_i) k'_{t+1}(\tilde{w}(m_i))(-\delta_1(\varepsilon)) + o(\varepsilon).
\end{aligned}$$

Since $k'_{t+1} < 0$ and under condition (33) $(\delta_3(\varepsilon) - \delta_1(\varepsilon)) \leq 0$, the expression above is strictly positive for ε small enough. This shows that $\left(u_v^0, w_v^0\right)$ cannot be optimal.

If $u_v^0(m_{n-2}) = u_v^0(m_{n-1})$, then the same steps as before go through if $u_v^0(m_i)$ is reduced by ε for all i such that $u_v^0(m_i) = u_v^0(m_{n-1})$ and δ are adjusted accordingly. ■

We are now ready to prove Proposition 4.

Proof of Proposition 4. The solution to Bellman equation (20) may involve randomization over several $\tilde{v}_v(z)$ that in expectations deliver v, but $\tilde{v}_v(z) \to \infty$ for all z as $v \to \infty$. Consider maximization problem for a given z and assume that agents send reports over the message set M_Θ for all v. Using homogeneity properties of $k_t(v)$, if (u'_v, w', σ_v) is a solution to $k_t(v)$, then $(u^x, w^x, \sigma^x) \equiv \left(v^{-1} \cdot u'_v, v^{-1} \cdot w'_v, \sigma_v\right)$ is a solution to the following problem for $x = v^{-1}$

$$\max_{u,w,\sigma} \mathbb{E}_\sigma \left[x^{a-1}\theta u - \zeta_t C(u) + \hat{\beta}_{t+1} k_{t+1}(w, x)\right] - x^a \chi_t W_t(\sigma)$$

subject to (22) and

$$1 = \mathbb{E}_\sigma [\theta u + \beta w].$$

Since u, w are bounded from below by 0, arguments analogous to (45) establish that $w^x(m) \in \left[0, (\pi(\theta_1)\beta)^{-1}\right]$ and that $u^x(m)$ lies in a compact set. Since Lemma 13 established that $k_{t+1}(w, x)$ is continuous in x, the Theorem of Maximum applies and solution correspondence (u^x, w^x, σ^x) is u.h.c. in x.

We show that there cannot be several types θ that send the same message m with a positive probability for low x, which establishes the result of the Proposition. First, observe that there must be some threshold $\bar{x}$, such that for all $x \leq \bar{x}$ no two types send the same message with probability 1. If this is not the case, we can choose a sequence $x_n \to 0$ with solution σ^{x_n} satisfying such property, which by u.h.c. of σ^{x_n} would imply that σ^0 satisfies this property, violating Lemma 15.

Next we rule out that several types send the same message with positive probability. Suppose that for any $\bar{x}$ we can find some $x < \bar{x}$ with this properties. There must be some type θ who is indifferent between messages m' and m''. In this case condition (30) holds and takes the form

$$\begin{aligned}
& \left[x^{a-1}\theta u^x(m') - \zeta_t C(u^x(m')) + \hat{\beta}_{t+1} k_{t+1}(w^x(m'))\right] \\
& - \left[x^{a-1}\theta u^x(m'') - \zeta_t C(u^x(m'')) + \hat{\beta}_{t+1} k_{t+1}(w^x(m''))\right] \\
= \quad & x^a \chi_t \left\{\left[\theta u^w(m'') - \lambda_t^w C(u^w(m''))\right] - \left[\theta u^w(m') - \lambda_t^w C(u^w(m'))\right]\right\}.
\end{aligned}$$

Since u^x, u^w lie in a compact set, taking sequence $x_n \to 0$ we get, invoking upper-hemicontinuity again,

$$\left[-\zeta_t C(u^0(m')) + \hat{\beta}_{t+1} k_{t+1}(w^0(m'))\right] - \left[-\zeta_t C(u^0(m'')) + \hat{\beta}_{t+1} k_{t+1}(w^0(m''))\right] = 0,$$

which violates Lemma 14. ■

8.5 Arguments for case with persistent types

In the following proofs we assume that the set of feasible utility is bounded and, without loss of generality, we set the lower bound to 0. Also, we assume that the message set M_t chosen by the planner is finite for all t. The proof that in the worst equilibrium there is no information revelation to the government is the same as in the iid case. The payoff of this equilibrium depends on the government's information that dissipates slowly due to the persistence of the shocks. Thus, the best payoff for a government that deviates at time t is

$$\tilde{W}_t(\mu_t) = \sup_{\{u_{t+s}(h)\}_{h\in H^t, s\geq 0}} \int_{H^t} \sum_{s=0}^{\infty} \beta^s \mathbb{E}_{\mathbf{p}_{t+s}}[\theta|h]\, u_{t+s}(h)\, d\mu_t \tag{66}$$

subject to the feasibility constraint (8) holding for all $t+s$ and

$$p_{t+s}\left(\theta|h^t\right) = \int_{\Theta} \pi^s\left(\theta|\theta^-\right) p_t\left(d\theta^-|h^t\right) \text{ for } s > 0.$$

This payoff is a generalization of (7) in the iid case. Similarly to that case, we can bound $\tilde{W}_t(\mu_t)$ with a function that is linear in μ_{t-1}. Given $p \in \Delta(\Theta)$, define the analogue of (13) as

$$W_t(\sigma, p) = \sup_{\{u_{t+s}(m)\}_{s\geq 0}} \int_{M_t\times\Theta\times\Theta} \sigma(dm|\theta)\, \pi\left(d\theta|\theta^-\right) p\left(d\theta^-\right) \sum_{s=0}^{\infty} \beta^s \begin{pmatrix} \int_{\Theta} \pi^s(d\theta_s|\theta)\, \theta_s u_{t+s}(m) \\ -\lambda^w_{t,t+s}\left(C\left(u_{t+s}(m)\right) - e\right) \end{pmatrix}. \tag{67}$$

As in the iid case, let $u^w_{t+s}(m)$ denote the solution to (67), which is given by equation $\mathbb{E}_{\mathbf{p}_{t+s}}[\theta|m] = \lambda^w_{t,t+s} C\left(u^w_{t+s}(m)\right)$.

By Langrange duality we can prove the analogue of Lemma 2 in the iid case.

Lemma 16 *The multiplier $\lambda^w_{t,t+s}$ is uniformly bounded away from 0 and belongs to a compact set for all t and s. Therefore, $u^w_{t+s}(m)$ belongs to a compact set in the interior of $[(1-\beta)\underline{v}, (1-\beta)\bar{v}]$ for all m, σ, t, s.*

Function W_t is well defined, continuous, convex in σ_t, uniformly bounded in t, and is minimized if and only if σ is uninformative. For any $\left(\sigma_t, p_t, \mu_{t-1}\right)$,

$$\tilde{W}_t(\mu_t) \leq \int_{\breve{H}^t\times Z} W_t\left(\sigma_t\left(\cdot|\breve{h}^t, \cdot\right), p_t\left(h^{t-1}\right)\right) d\mu_{t-1} dz, \tag{68}$$

with equality if $\left(\sigma_t, p_t, \mu_{t-1}\right) = \left(\sigma^*_t, \hat{p}^*_t, \mu^*_{t-1}\right)$.

Proof. The objective function (66) is concave and the constraint set is convex and we can use the Lagrange duality and rewrite $\tilde{W}_t(\mu_t)$ as

$$\tilde{W}_t(\mu_t) = \min_{\{\lambda_{t,s}\geq 0\}_{s\geq 0}} \sup_{\{u_{t+s}(h)\}_{h\in H^t, s\geq 0}} \left\{ \begin{array}{c} \int_{H^t\times Z} \sum_{s=0}^{\infty} \beta^s \mathbb{E}_{\mathbf{p}_{t+s}}[\theta|h]\, u_{t+s}(h)\, d\mu_t dz \\ -\sum_{s=0}^{\infty} \beta^s \lambda_{t,t+s}\left(\int_{H^t\times Z} C\left(u_{t+s}(h)\right) d\mu_t dz - e\right) \end{array} \right\}. \tag{69}$$

Let $\lambda^w_{t,t+s}$ be the solution to the minimization problem. Since after deviating, the government doesn't provide incentives, we can maximize $\tilde{W}_t$ period by period.

$$\sup_{\{u_{t+s}(h)\}_{h\in H^t, s\geq 0}} \int_{H^t\times Z} \beta^s \left(\mathbb{E}_{\mathbf{p}_{t+s}}[\theta|h]\, u_{t+s}(h) - \lambda^w_{t,t+s} C\left(u_{t+s}(h)\right) + \lambda^w_{t,t+s} e\right) d\mu_t dz.$$

By using the same argument as in the iid case we can show that $\lambda^w_{t,t+s}$ is uniformly bounded away from 0 and uniformly bounded above for all s. This also proves that supremum in (69) is achieved. Also,

$$\begin{aligned}
\tilde{W}_t(\mu_t) &= \min_{\{\lambda_{t,s}\geq 0\}_{s\geq 0}} \max_{\{u_{t+s}(h)\}_{h\in H^t, s\geq 0}} \left\{ \begin{array}{c} \int_{H^t\times Z} \sum_{s=0}^{\infty} \beta^s \mathbb{E}_{\mathbf{p}_{t+s}}[\theta|h]\, u_{t+s}(h)\, d\mu_t dz \\ -\sum_{s=0}^{\infty} \beta^s \lambda_{t,t+s} \left(\int_{H^t\times Z} C(u_{t+s}(h))\, d\mu_t dz - e \right) \end{array} \right\} \\
&\leq \max_{\{u_{t+s}(h)\}_{h\in H^t, s\geq 0}} \int_{H^t\times Z} \sum_{s=0}^{\infty} \beta^s \left(\mathbb{E}_{\mathbf{p}_{t+s}}[\theta|h]\, u_{t+s}(h) - \lambda^w_{t,t+s}\left(C(u_{t+s}(h)) - e\right)\right) d\mu_t dz
\end{aligned}$$

where the inequality follows from the fact that $\{\lambda^w_{t,t+s}\}$ may not be a minimizer for an arbitrary $\left(\sigma_t, p_t, \mu_{t-1}\right)$. Since

$$\max_{\{u_{t+s}(h)\}_{h\in H^t, s\geq 0}} \int_{H^t\times Z} \sum_{s=0}^{\infty} \beta^s \left(\mathbb{E}_{\mathbf{p}_{t+s}}[\theta|h]\, u_{t+s}(h) - \lambda^w_{t,t+s}\left(C(u_{t+s}(h)) - e\right)\right) d\mu_t dz$$

$$= \int_{H^{t-1}\times Z} \max_{\{u_{t+s}(m)\}_{s\geq 0}} \int_{M_t\times\Theta\times\Theta} \sigma_t(dm|\theta)\, \pi\left(d\theta|\theta^-\right) p_t\left(d\theta^-\right) \sum_{s=0}^{\infty} \beta^s \left[\begin{array}{c} \left(\int_\Theta \pi^s(d\theta_s|\theta)\, \theta_s \right) u_{t+s}(m) \\ -\lambda^w_{t,t+s}\left(C(u_{t+s}(m)) - e\right) \end{array} \right] d\mu_{t-1} dz$$

equation (68) must hold.

The same arguments show that we can maximize (67) period by period, thus, if we let

$$\hat{W}_{t+s}(\sigma_t, p_t) = \max_{\{u_{t+s}(m)\}} \int_{M_t\times\Theta\times\Theta} \sigma_t(dm|\theta)\, \pi\left(d\theta|\theta^-\right) p_t\left(d\theta^-\right) \left(\begin{array}{c} \int_\Theta \pi^s(d\theta_s|\theta)\, \theta_s u_{t+s}(m) \\ -\lambda^w_{t,t+s}\left(C(u_{t+s}(m)) - e\right) \end{array} \right),$$

the optimal $u^w_{t+s}(m)$ satisfies

$$C'(u_{t+s}(m)) = \frac{1}{\lambda^w_{t,t+s}} \int_\Theta p\left(d\theta^-\right) \mathbb{E}_\sigma\left[\theta|m, \theta^-\right] \in \left[\frac{\theta_1}{\lambda^w_{t,t+s}}, \frac{\theta_{|\Theta|}}{\lambda^w_{t,t+s}} \right],$$

which implies that $\hat{W}_{t+s}$ is uniformly bounded in $t+s$, σ_t, and p_t and that we can restrict $u_{t+s}(m)$ to a compact set. The latter implies continuity of $\hat{W}_{t+s}$ by the Theorem of the Maximum. Finally, since $\hat{W}_{t+s}$ is uniformly bounded, we have that $W_t(\sigma_t, p_t) = \sum_{s=0}^{\infty} \beta^s \hat{W}_{t+s}(\sigma_t, p_t)$ is also bounded and continuous.

The proof that W_t is convex in σ_t follows exactly the same steps as the in the iid case.

To see that W_t is minimized at an uninformative signal, let $\bar{\sigma}$ be uninformative and note that for any σ

$$\begin{aligned}
W_t(\sigma_t, p_t) &= \max_{\{u_{t+s}(m)\}_{s\geq 0}} \int_{M_t\times\Theta\times\Theta} \sigma_t(dm|\theta)\, \pi\left(d\theta|\theta^-\right) p_t\left(d\theta^-\right) \sum_{s=0}^{\infty} \beta^s \left[\begin{array}{c} \left(\int_\Theta \pi^s(d\theta_s|\theta)\, \theta_s \right) u_{t+s}(m) \\ -\lambda^w_{t,t+s}\left(C(u_{t+s}(m)) - e\right) \end{array} \right] \\
&\geq \max_{\{\bar{u}_{t+s}\}_{s\geq 0}} \int_{\Theta\times\Theta} \pi\left(d\theta|\theta^-\right) p_t\left(d\theta^-\right) \sum_{s=0}^{\infty} \beta^s \left[\theta_s \bar{u}_{t+s} - \lambda^w_{t,t+s}\left(C(\bar{u}_{t+s}) - e\right)\right] \\
&= W_t(\bar{\sigma}_t, p_t)
\end{aligned}$$

Let $\mu(A) = \int_\Theta \sigma(A|\theta)\,\pi(d\theta)$ for any Borel A of message set M_t and let $\bar{u}^w_{t+s}$ be the optimal solution when the strategy is uninformative, then

$$\begin{aligned}
& W_t(\sigma_t, p_t) - W_t(\bar{\sigma}_t, p_t) \\
= \; & \max_{\{u_{t+s}(m)\}_{s\geq 0}} \int\limits_{M_t\times\Theta\times\Theta} \sigma_t(dm|\theta)\,\pi\left(d\theta|\theta^-\right) p_t\left(d\theta^-\right) \\
& \sum_{s=0}^{\infty} \beta^s \left[\left(\int\limits_\Theta \pi^s(d\theta_s|\theta)\,\theta_s \right) u_{t+s}(m) - \lambda^w_{t,t+s}\left(C(u_{t+s}(m)) - e\right) - \left(\bar{u}^w_{t+s} - \lambda^w_{t,t+s}\left(C\left(\bar{u}^w_{t+s}\right) - e\right)\right) \right]
\end{aligned}$$

The expression in square bracket is non-negative. Moreover, if σ_t is informative, then there is a set of messages A with $\mu(A) > 0$ such that $|\mathbb{E}_\sigma[\theta|A] - 1| > 0$. For all such messages the expression in square brackets is strictly positive since $\bar{u}^w_t$ does not satisfy the optimality condition

$$C'(u_t(m)) = \frac{1}{\lambda^w_{t,t}} \mathbb{E}_\sigma[\theta|A]$$

for $m \in A$. Since $\mu(A) > 0$, $W_t(\sigma_t, p_t) - W_t(\bar{\sigma}_t, p_t)$ is strictly positive and, hence, any informative σ cannot be a minimum. ■

Similarly to the iid case, we replace the incentive constraint for the government $\mathbb{E}_{\boldsymbol{\sigma}} \sum_{t=s}^{\infty} \beta^{t-s}\theta_t u_t \geq \tilde{W}_t(\mu_t)$ with

$$\mathbb{E}_{\boldsymbol{\sigma}} \sum_{t=s}^{\infty} \beta^{t-s}\theta_t u_t \geq \int_{\breve{H}^t\times Z} W_t\left(\sigma_t\left(\cdot|\breve{h}^t, \cdot\right), p_t\left(h^{t-1}\right)\right) d\mu_{t-1} dz \tag{70}$$

and use standard techniques to derive the analogue of the Langrangian (17) for the persistent case. The problem becomes

$$\mathcal{L} = \max_{\mathbf{u},\boldsymbol{\sigma}} \mathbb{E}_{\boldsymbol{\sigma}} \sum_{t=0}^{\infty} \bar{\beta}_t \left[\theta_t u_t - \zeta_t C(u_t) - \chi_t W_t\right] \tag{71}$$

subject to

$$p_t\left(\theta|h^t\right) = \frac{\int_\Theta \sigma_t\left(m|\breve{h}^t, \theta\right) \pi\left(\theta|\theta^-\right) p_{t-1}\left(d\theta^-|h^{t-1}\right)}{\int_{\Theta\times\Theta} \sigma_t\left(m|\breve{h}^t, \theta\right) \pi\left(d\theta|\theta^-\right) p_{t-1}\left(d\theta^-|h^{t-1}\right)},$$

for all $h^{t-1}, \breve{h}^t$ and m, whenever defined, the incentive constraint

$$\mathbb{E}_{\boldsymbol{\sigma}}\left[\sum_{t=0}^{\infty} \beta^t\theta u \middle| \theta^-\right] \geq \mathbb{E}_{\boldsymbol{\sigma}'}\left[\sum_{t=0}^{\infty} \beta^t\theta u \middle| \theta^-\right] \text{ for all } \boldsymbol{\sigma}', \theta^-, \tag{72}$$

and

$$\mathbb{E}_{\boldsymbol{\sigma}}\left[\sum_{t=0}^{\infty} \beta^t\theta u \middle| v\right] = v, \tag{73}$$

for some non-negative sequences of Lagrange multipliers $\{\bar{\beta}_t, \chi_t, \zeta_t\}_{t=0}^{\infty}$ with the property that $\hat{\beta}_t \equiv \bar{\beta}_t/\bar{\beta}_{t-1} \geq \beta$ with strict inequality if and only if (70) binds in period t.

Finally, we adapt the arguments in Fernandes and Phelan (2000) with minor modifications and obtain a recursive representation of (71). In particular, we first rewrite the constranit set in a recursive form by adding the promise-keeping constraint (34) for each type θ^-. Since it is no longer optimal for agents to always report their types truthfully, unlike in the case studied in Fernandes and Phelan (2000), the agent' previous type, θ^-, cannot be used as a state variable. Instead, the recursive formulation keeps track of the planner's posterion beliefs about θ^-. Therefore,

$$k_t(\mathbf{v}, p) = \max_{\substack{\{u,w,\sigma\} \\ \sigma(\cdot|\theta)\in\Delta(M_\Theta),p'\in\Delta(\Theta)}} \int_\Theta p\left(d\theta^-\right) \mathbb{E}_\sigma \left[\theta u - \zeta_t C(u) + \hat{\beta}_{t+1} k_{t+1}\left(\mathbf{w}, p'\right) |\theta^-\right] - \chi_t \int W_t dz \tag{74}$$

subject to (34),

$$p'(\theta|m,z) = \frac{\int_\Theta \sigma(m|\theta,z)\,\pi\left(\theta|\theta^-\right) p\left(d\theta^-\right)}{\int_{\Theta\times\Theta} \sigma(m|\theta,z)\,\pi\left(d\theta|\theta^-\right) p\left(d\theta^-\right)}$$

whenever defined, the incentive constraint

$$\mathbb{E}_\sigma\left[\theta u + \beta w(\cdot,\cdot,\theta)\,|\theta,z\right] \geq \mathbb{E}_{\sigma'}\left[\theta u + \beta w(\cdot,\cdot,\theta)\,|\theta,z\right] \text{ for all } z,\theta,\sigma'. \tag{75}$$

We now extend the arguments of the simple example in Section 4.1 to the persistent case. As in the iid case, the assumption that the government can deviate at time $t \geq 1$ only if it deviates at $t = 0$ implies that the Lagrangian (71) can be written as

$$\mathcal{L} = \max_{\mathbf{u},\boldsymbol{\sigma}} \mathbb{E}_{\boldsymbol{\sigma}} \left\{ -\sum_{t=0}^{\infty} \beta^t \zeta_t C(u_t) - \chi_0 W_0 \right\}$$

subject to (72) and (73). Consider the subgame starting from $t = 1$. From past reports, the planner has beliefs $p \in \Delta(\Theta)$. Also, as in the iid case, since the sustainability constraint does not bind at $t \geq 1$, the standard Revelation principle applies and the agent will report his type truthfully (let m_θ denote the message corresponding to type θ.) Thus, as in Fernandes and Phelan (2000), from time $t \geq 2$ we can replace the posterior of the planner with the agent's type. At time 1, let $\kappa_1(\mathbf{v}, p)$ be defined as

$$\kappa_1(\mathbf{v}, p) = \max_{\{u(m),w(m,\theta)\}_{m\in M_1,\theta\in\Theta}} \sum_i p\left(\theta_i^-\right) \mathbb{E}\left[-\zeta_1 C(u) + \beta\kappa_2(\mathbf{w};\theta)\,|\theta_i^-\right]$$

subject to

$$v\left(\theta_i^-\right) = \sum_{m,\theta} \pi\left(\theta|\theta_i^-\right)\left[\theta u(m_\theta) + \beta w(m_\theta,\theta)\,|\theta_i^-\right],\ \forall i$$

and

$$\theta u(m_\theta) + \beta w(m_\theta, \theta) \geq \theta u\left(m'_\theta\right) + \beta w\left(m'_\theta, \theta\right), \ \forall m_\theta, m'_\theta, \theta.$$

The constraints are linear in $(u, \mathbf{w})$, hence, if $(u, \mathbf{w})$, $\kappa_2(\mathbf{w}; \theta)$, and $\kappa_1(\mathbf{v}, p)$ solve the Bellman equation for some $(\mathbf{v}, p)$, it is immediate to see that $(x\ u, x\ \mathbf{w})$, $x^a \kappa_2(\mathbf{w}, \theta)$, and $x^a \kappa_1(\mathbf{v}, p)$ also solve the Bellman equation for $(x\ \mathbf{v}, p)$, $x > 0$. Thus, $\kappa_1(\mathbf{v}, p)$ is homogenous in $\mathbf{v}$. Also, since the objective function is linear in p and the constraints are independent of p, $\kappa_1(\mathbf{v}, p)$ is convex in p.

At time 0, for any reporting strategy $\sigma \in \Delta(M_0)$, let $\kappa_0(\mathbf{v}; \sigma)$ be defined as

$$\kappa_0(\mathbf{v}; \sigma) = \max_{\{u(m), w(m,\theta)\}_{m \in M_0, \theta \in \Theta}} \mathbb{E}_\sigma\left[-\zeta_0 C(u) + \beta \kappa_1(\mathbf{w}, p)\right]$$

subject to

$$p(\theta|m) = \frac{\sigma(m|\theta) \sum_{\theta_- \in \Theta} \pi\left(\theta|\theta^-\right) \bar{\pi}\left(\theta^-\right)}{\sum_{(\theta,\theta_-) \in \Theta^2} \sigma(m|\theta) \pi\left(\theta|\theta^-\right) \bar{\pi}\left(\theta^-\right)}$$

whenever defined, (34), and (75), where $\bar{\pi}$ is the planner's initial prior. Again, constraints are linear in $(u, \mathbf{w})$ and $\kappa_1(\mathbf{v}, p)$ is homogenous in $\mathbf{v}$, thus, $\kappa_0(\mathbf{v}; \sigma)$ will also be homogenous in $\mathbf{v}$. If $\sigma = \sigma^{un}$ is uninformative and $\Pr(m) = \sum_{\theta \in \Theta} \sigma^{un}(m|\theta) \sum_{\theta_- \in \Theta} \pi\left(\theta|\theta^-\right) \bar{\pi}\left(\theta^-\right) > 0$, then

$$p(\theta|m) = \sigma^{un}(m|\theta) \frac{\sum_{\theta_- \in \Theta} \pi\left(\theta|\theta^-\right) \bar{\pi}\left(\theta^-\right)}{\Pr(m)} = \sum_{\theta_- \in \Theta} \pi\left(\theta|\theta^-\right) \bar{\pi}\left(\theta^-\right).$$

and the optimal allocation $(u^{un}, \mathbf{w}^{un})$ is $u^{un}(m) = \bar{u}$ and $w^{un}(m, \theta) = \bar{w}(\theta)$ for all θ. Suppose instead that σ is informative, then there exist m', m'', and θ such that $p^{in}(\theta|m') \neq p^{in}(\theta|m'')$. By convexity of κ_1 in p,

$$\mathbb{E}_\sigma\left[\kappa_1\left(\mathbf{w}^{un}, p^{in}\right)\right] \geq \mathbb{E}_\sigma\left[\kappa_1\left(\mathbf{w}^{un}, p^{un}\right)\right].$$

Finally, if the distribution of types is such that no bunching is desirable under commitment, the latter implies $\kappa_0(\mathbf{v}; \sigma) > \kappa_0(\mathbf{v}; \sigma^{un})$.

To prove Proposition 5 we need the following Lemma.

Lemma 17 $\Pr(u_{\mathbf{v},p}(m, z) \to 0) \to 1$ *and* $\Pr(w_{\mathbf{v},p}(m, z, \theta) \to 0) \to 1$ *as* $\mathbf{v} \to 0$*, for all* m, θ*, and* p*.*

Proof. From (34) for all p we have

$$\begin{aligned} v\left(\theta^{-}\right) &= \int_{\Theta\times M_t\times Z}\sigma_{\mathbf{v},p}\left(dm|\theta\right)\pi\left(d\theta|\theta^{-}\right)\left[\theta u_{\mathbf{v},p}\left(m,z\right)+\beta w_{\mathbf{v},p}\left(m,z,\theta\right)\right]dz \\ &= \int_{\Theta\times Z}\pi\left(d\theta|\theta^{-}\right)\left[\theta u_{\mathbf{v},p}\left(m,z\right)+\beta w_{\mathbf{v},p}\left(m,z,\theta\right)\right]dz,\ m\in Supp\left(\sigma_{\mathbf{v},p}\left(\cdot|\theta\right)\right) \\ &\geq \int_{\Theta\times Z}\pi\left(d\theta|\theta^{-}\right)\left[\theta u_{\mathbf{v},p}\left(\tilde{m},z\right)+\beta w_{\mathbf{v},p}\left(\tilde{m},z,\theta\right)\right]dz,\ \forall\tilde{m} \\ &\geq \int_{Z}\left[\theta u_{\mathbf{v},p}\left(\tilde{m},z\right)+\beta w_{\mathbf{v},p}\left(\tilde{m},z,\theta\right)\right]dz,\ \forall\tilde{m},\theta. \\ &\geq \beta\int_{Z}w_{\mathbf{v},p}\left(m,z,\theta\right)dz,\ \forall m,\theta. \end{aligned}$$

Here the fourth and fifth lines follow from nonnegativity of u and $\mathbf{w}$, and the third line follows from (75). Thus, $\lim_{\mathbf{v}\to\mathbf{0}}\int_Z w_{\mathbf{v},p}\left(m,z,\theta\right)dz=0,\ \forall m,\theta,p$. Analogous arguments prove that $\lim_{v\to 0}\int_Z u_{\mathbf{v},p}\left(m,z\right)dz=0,\ \forall m,p$. Since $u_{\mathbf{v},p}$ and $\mathbf{w}_{\mathbf{v},p}$ are bounded below by 0, the latter implies that $\Pr\left(u_{\mathbf{v},p}\left(m,z\right)\to 0\right)\to 1$ and $\Pr\left(w_{\mathbf{v},p}\left(m,z,\theta\right)\to 0\right)\to 1$ for all m,θ, and p. ■

Proof of Proposition 5. Let $\bar{\sigma}$ be any uninformative strategy. Let

$$\bar{u}_{\mathbf{v},p}\left(m,z\right)=\int_{\Theta\times M_t}\pi\left(d\theta|\theta^{-}\right)\bar{\pi}\left(d\theta^{-}\right)\sigma_{\mathbf{v},p}\left(dm|\theta\right)\theta u_{\mathbf{v},p}\left(m,z\right)$$

and

$$\bar{w}_{\mathbf{v},p}\left(m,z,\theta\right)=\frac{1}{\beta}\int_{M_t}\sigma_{\mathbf{v},p}\left(dm|\theta\right)\left[\theta u_{\mathbf{v},p}\left(m,z\right)+\beta w_{\mathbf{v},p}\left(m,z,\theta\right)\right]-\frac{1}{\beta}\theta\bar{u}_{\mathbf{v},p}\left(m,z\right).$$

Profile $\left(\bar{u}_{\mathbf{v},p},\bar{\mathbf{w}}_{\mathbf{v},,p},\bar{\sigma}\right)$ is incentive compatible and satisfies

$$\begin{aligned} &\int_{\Theta\times M_t\times Z}\sigma_{\mathbf{v},p}\left(dm|\theta\right)\pi\left(d\theta|\theta^{-}\right)\left[\theta\bar{u}_{\mathbf{v},p}\left(m,z\right)+\beta\bar{w}_{\mathbf{v},p}\left(m,z,\theta\right)\right]dz \\ =&\int_{\Theta\times Z}\pi\left(d\theta|\theta^{-}\right)\left[\theta\bar{u}_{\mathbf{v},p}\left(m,z\right)+\beta\bar{w}_{\mathbf{v},p}\left(m,z,\theta\right)\right]dz \\ =&\int_{\Theta\times Z}\pi\left(d\theta|\theta^{-}\right)\left(\theta\bar{u}_{\mathbf{v},p}\left(m,z\right)+\int_{M_t}\sigma_{\mathbf{v},p}\left(dm|\theta\right)\left[\theta u_{\mathbf{v},p}\left(m,z\right)+\beta w_{\mathbf{v},p}\left(m,z,\theta\right)\right]-\theta\bar{u}_{\mathbf{v},p}\left(m,z\right)\right)dz \\ =&\int_{\Theta\times M_t\times Z}\sigma_{\mathbf{v},p}\left(dm|\theta\right)\pi\left(d\theta|\theta^{-}\right)\left[\theta u_{\mathbf{v},p}\left(m,z\right)+\beta w_{\mathbf{v},p}\left(m,z,\theta\right)\right]dz=v\left(\theta^{-}\right) \end{aligned}$$

Therefore the value of the objective function (74) evaluated at $\left(u_{\mathbf{v},p},\mathbf{w}_{\mathbf{v},p},\sigma_{\mathbf{v},p}\right)$ should be higher than evaluated at $\left(\bar{u}_{\mathbf{v},p},\bar{\mathbf{w}}_{\mathbf{v},p},\bar{\sigma}\right)$,

$$\begin{aligned} &\int_{\Theta\times Z}p\left(d\theta^{-}\right)\mathbb{E}_{\sigma_{\mathbf{v},p}}\left[\begin{array}{c}\theta u_{\mathbf{v},p}-\zeta_t C\left(u_{\mathbf{v},p}\right)+\hat{\beta}_{t+1}k_{t+1}\left(\mathbf{w}_{\mathbf{v},p},p'\right)\\ -\chi_t W_t\left(\sigma_{\mathbf{v},p},p\right)\end{array}\middle|\theta^{-}\right]dz \qquad (76) \\ \geq&\int_{\Theta\times Z}p\left(d\theta^{-}\right)\mathbb{E}_{\sigma_{\mathbf{v},p}}\left[\theta\bar{u}_{\mathbf{v},p}-\zeta_t C\left(\bar{u}_{\mathbf{v},p}\right)+\hat{\beta}_{t+1}k_{t+1}\left(\bar{\mathbf{w}}_{\mathbf{v},p},p'\right)-\chi_t W_t\left(\bar{\sigma},p\right)|\theta^{-}\right]dz \end{aligned}$$

From Lemma 17, $\Pr\left(u_{\mathbf{v},p}\left(m,z\right)\rightarrow 0\right)\rightarrow 1$ and $\Pr\left(w_{\mathbf{v},p}\left(m,z,\theta\right)\rightarrow 0\right)\rightarrow 1$ for all m,θ, and p and, thus, $\Pr\left(\bar{u}_{\mathbf{v},p}\left(m,z\right)\rightarrow 0\right)\rightarrow 1$ and $\Pr\left(\bar{w}_{\mathbf{v},p}\left(m,z,\theta\right)\rightarrow 0\right)\rightarrow 1$ for all m,θ, and p. Also, similarly to the iid case, Lemma 3 in Acemoglu, Golosov and Tsyvinski (2008) implies that it is sufficient to randomize among only a finite number of points $\{u_{\mathbf{v},p},w_{\mathbf{v},p},\sigma_{\mathbf{v},p}\}$. Therefore, for each p

$$\int_{\Theta\times M_t\times Z}\sigma_{\mathbf{v},p}\left(dm|\theta\right)\pi\left(d\theta|\theta^-\right)p\left(d\theta^-\right)\left[\begin{array}{c}\theta\left(u_{\mathbf{v},p}\left(m\right)-\bar{u}_{\mathbf{v},p}\left(m\right)\right)-\zeta_t\left(C\left(u_{\mathbf{v},p}\left(m\right)\right)-C\left(\bar{u}_{\mathbf{v},p}\left(m\right)\right)\right)\\+\hat{\beta}_{t+1}\left(k_{t+1}\left(\mathbf{w}_{\mathbf{v},p},p'\right)-k_{t+1}\left(\mathbf{w}_{\mathbf{v},p},p'\right)\right)\end{array}\right]dz$$
$$-\chi_t\int_Z\left(W_t\left(\sigma_{\mathbf{v},p},p\right)-W_t\left(\bar{\sigma},p\right)\right)dz$$

converges to $-\chi_t\int_Z\left(W_t\left(\sigma_{\mathbf{v},p},p\right)-W_t\left(\bar{\sigma},p\right)\right)dz$. Hence, in the limit equation (76) becomes

$$\limsup_{\mathbf{v}\rightarrow 0}\chi_t\int_Z\left(W_t\left(\bar{\sigma},p\right)-W_t\left(\sigma_{\mathbf{v},p},p\right)\right)dz\geq 0,$$

for all p. Since $W_t\left(\sigma_{\mathbf{v},p},p\right)\geq W_t\left(\bar{\sigma},p\right)$ by Lemma 16 and $\chi_t>0$ by assumption, $\Pr\left(W_t\left(\sigma_{\mathbf{v},p},p\right)\rightarrow W_t\left(\bar{\sigma},p\right)\right)\rightarrow 1$ as $\mathbf{v}\rightarrow 0$ for all p. By Lemma 16, $W_t\left(\sigma,p\right)$ is continuous in σ and achieves its minimum only at uninformative reporting strategies, therefore $\Pr\left(\sigma_{\mathbf{v},p}\rightarrow\bar{\sigma}\right)\rightarrow 1$ for all p. ■