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Comprehensive Exam

1. a) Formally define what it means for a test to control i) the (asymptotic) null rejection probability and ii) the (asymptotic) size. Provide an example of a test where for a certain parameter space the asymptotic null rejection probability is controlled but not the asymptotic size. What is the issue for applied researchers when employing such a test?

   b) In a scenario where there are several tests that control the size and one of them is uniformly most powerful (UMP) applied researchers should use that test. However, often UMP tests do not exists. a) Give a (nontrivial) example where there exists a UMP test. Prove your claim. b) In a situation where no UMP test exists (in the class of tests that control size) describe general criteria according to which one could still potentially single out a recommended test. Provide an example of that strategy (with details) that we dealt with in class.

2. Under Assumptions EE3 and CF-NS (stated below) we derived the limiting distribution of the GMM estimator $\widehat{\theta}_n$ with nonsmooth stochastic criterion function, namely

$$\sqrt{n}(\widehat{\theta}_n - \theta_0) \to_d N(0, (\Gamma'\Gamma)^{-1}\Gamma' V_0 \Gamma(\Gamma'\Gamma)^{-1}).$$

a) Provide estimators for V_0 and Γ and discuss their consistency under appropriate conditions.

Consider the example of Quantile Regression, $(Y_i, X_i')'$ i.i.d.

$$Y_i = X_i'\theta_0 + U_i,$$

where the τ-quantile of U_i conditional on X_i equals 0 for some $\tau \in (0, 1)$.

b) Derive what V_0 and Γ are in this case.

c) Find a simple expression for $(\Gamma'\Gamma)^{-1}\Gamma' V_0 \Gamma(\Gamma'\Gamma)^{-1}$ in this case.

Setup and Assumptions for the GMM case with nonsmooth stochastic criterion function:

Let $Q_n(\theta) = ||\overline{g}_n(\theta)||$, where $\overline{g}_n(\theta) = n^{-1}\sum_{i=1}^n g(W_i, \theta)$, and $g(\theta) = Eg(W_i, \theta)$.

Assumption CF-NS: (i) *θ_0 is in the interior of Θ.*

(ii) *$g(\theta)$ is differentiable at θ_0 with* $\Gamma = (\partial/\partial\theta')g(\theta_0)$ of full rank $d \leq k$.

(iii) $g(\theta_0) = 0$.

(iii) $\sqrt{n}\overline{g}_n(\theta_0) \to_d N(0, V_0)$.

(iv) For every sequence of positive constants $\{\delta_n\}_{n\geq 1}$ that converges to zero,

$$\sup_{\theta\in\Theta, ||\theta-\theta_0||<\delta_n} \sqrt{n}\,||\overline{g}_n(\theta) - g(\theta) - \overline{g}_n(\theta_0)|| \to_p 0.$$

Assumption EE3: (i) $Q_n(\widehat{\theta}_n) = \inf_{\theta\in\Theta} Q_n(\theta) + o_p(n^{-1/2})$ and (ii) $\widehat{\theta}_n \to_p \theta_0$.

3. a) Consider the model defined by the equations

$$\begin{aligned} y &= x'\beta + z_1'\gamma + \varepsilon, \\ x &= \Pi z + \eta, \end{aligned}$$

where $z = (z_1', z_2')' \in R^{d_z}$, $z_j \in R^{d_{z_j}}$ for $j = 1, 2$, $Ez\varepsilon = 0$, $Ez\eta' = 0$, and Ezz' is nonsingular. Show that $\theta = (\beta', \gamma')'$ is point identified under the rank condition $rk(\Pi_2) = d_x$, where $\Pi = (\Pi_1 : \Pi_2) \in R^{d_x \times d_z}$ and $\Pi_j \in R^{d_x \times d_{z_j}}$, for $j = 1, 2$.

b) Define the nonparametric selection model of Das, Newey, and Vella (2003) discussed in class. Without proof, specify conditions that are sufficient for identification of the derivative of the unknown function in the main equation.

4. Consider the model $Y = X'\beta + e$, $\beta \in R^p$, $E(e|X) = 0$ with *iid* observations.

a) Show that the bias and covariance matrix of the ridge estimator

$$\widehat{\beta}_{ridge} = (X'X + \lambda I_p)^{-1} X'Y$$

with fixed parameter $\lambda > 0$ are given by

$$\begin{aligned} bias(\widehat{\beta}_{ridge}|X) &= -\lambda(X'X + \lambda I_p)^{-1}\beta \\ var(\widehat{\beta}_{ridge}|X) &= (X'X + \lambda I_p)^{-1}(X'DX)(X'X + \lambda I_p)^{-1}, \end{aligned}$$

respectively, where $D := E(ee'|X)$.

b) Show that $mse(\widehat{\beta}_{ridge}|X) < mse(\widehat{\beta}_{OLS}|X)$ if $2\lambda_{\min}(D) > \lambda\beta'\beta$.

c) Discuss what is meant by the "oracle property" of the Lasso estimator. Critically discuss the underlying assumptions needed to obtain that result.

Comprehensive exam August 2023 — Joris Pinkse

The eight questions below should take less than ten minutes each. None require an answer that is more than a few lines long, so by writing long answers you are wasting your time.

1. Using the definition of a probability measure, show that $\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cup B)$.
2. Let $\boldsymbol{y} = \boldsymbol{x}^2$, where $\boldsymbol{x} \sim N(\mu, \sigma^2)$ for $\mu \in \mathbb{R}$ and $\sigma > 0$. For what values of μ, σ^2, (a) are $\boldsymbol{x}, \boldsymbol{y}$ uncorrelated, (b) is $\boldsymbol{x}$ conditionally mean independent of $\boldsymbol{y}$, (c) is $\boldsymbol{y}$ conditionally mean independent of $\boldsymbol{x}$, (d) are $\boldsymbol{x}, \boldsymbol{y}$ independent?
3. Let $\{X_i\}$ be matrix-valued i.i.d. variables with bounded support and mean M_0 and let $\bar{X}$ be their sample mean. Derive the limit distribution of $r_n\{\exp(u^\top \bar{X} v) - \exp(u^\top M_0 v)$ for any given vectors $u, v \neq 0$ and where r_n is such that the limit distribution exists and is not degenerate.
4. Suppose that $\theta_0 = \arg\min_{\theta \in \Theta} \Omega(\theta)$ (so θ_0 is unique) where Ω is continuous and Θ compact. Suppose further that $\sup_{\theta \in \Theta} |\hat{\Omega}(\theta) - \Omega(\theta)| = o_p(1)$. Prove that $\hat{\theta} = \arg\min_{\theta \in \Theta} \hat{\Omega}(\theta) \xrightarrow{p} \theta_0$.
5. Let $\boldsymbol{x}$ be a continuously distributed random variable with support $[1, \infty)$ and density function f. Let $\boldsymbol{y}_n = (1 + \boldsymbol{x}/n)^{-n}$ for all $n \in \mathbb{N}$. Prove that $\lim_{n \to \infty} \mathbb{E}\boldsymbol{y}_n$ exists and determine its limit.
6. Recall that the Hoeffding inequality says that if $\{\boldsymbol{x}_i\}$ is independent (but not necessarily identically distributed) and for some numbers $\{a_i\}$, $\{b_i\}$, $\Pr(a_i \leq \boldsymbol{x}_i \leq b_i) = 1$ then
$$\forall z > 0 : \Pr\left(\left|\sum_{i=1}^n (\boldsymbol{x}_i - \mathbb{E}\boldsymbol{x}_i)\right| > z\right) \leq 2\exp\left(\frac{-2z^2}{\sum_{i=1}^n (b_i - a_i)^2}\right).$$
Suppose that the support of the i.i.d. $\{\boldsymbol{x}_i\}$ is $[-1/\sqrt{2}, 1/\sqrt{2}]$. Construct a test for the hypothesis $H_0 : \mu_0 = 0$ against the alternative hypothesis $H_1 : \mu_0 \neq 0$ where $\mu_0 = \mathbb{E}\boldsymbol{x}_1$ that has rejection probability no greater than α under H_0. Use a decision rule of the form: 'reject H_0 if $|\bar{\boldsymbol{x}}| > C_\alpha$.'
7. Consider $\boldsymbol{x} = \boldsymbol{z}^\top W \boldsymbol{z}$ for a generic positive semidefinite matrix W, where $\boldsymbol{z}$ is a vector of i.i.d. $N(0,1)$ variables. Derive the exact distribution of $\boldsymbol{x}$.
8. Suppose that $\{\boldsymbol{x}_i\}$ is i.i.d. with density function $(6\sigma_0\sqrt{3})\pi^{-1}\{3\sigma_0^2 + (x - \mu_0)^2\}^{-1}$. Propose a consistent and asymptotically valid test for $H_0 : \sigma_0^2 = 1$ versus $H_1 : \sigma_0^2 \neq 1$. Please be precise, but you do not need to establish or argue consistency or validity.