Macro Qualifier Exam

Penn State: August, 2023

- You have 3.5 hours. There are four (multi-part) questions, one for each quarter of the first-year macro sequence. Each question is worth 45 points, so if you progress at a rate of one point per minute, you will be able to complete the exam with some time to spare.
- Neither books nor notes are permitted.
- If you make any assumptions beyond what's in the text of the question, please state those assumptions clearly.
- If you need more space, please ask for additional sheets of paper. If you use more sheets, please number the pages, write your identifying number in lieu of your name, and label clearly which question you are answering.
- Please write clearly. Show intermediate work for partial credit. Unannotated scratch work will receive no credit.

Good luck!

Quarter 1.

Consider an infinite-horizon economy with two infinitely-lived agents, agent 1 and 2. Each agent is endowed with a Lucas tree at $t = 0$. A Lucas tree lives forever and yields dividend in terms of the non-storable consumption good each period. The dividend of agent i's tree at t, d_t^i, is an i.i.d. random variable: $d_t^i \in \{e_h, e_l\}$, with $e_h > e_l > 0$,

$$\text{Prob}\{d_t^i = e_h\} = \mu, \qquad \text{Prob}\{d_t^i = e_l\} = 1 - \mu$$

$\mu \in (0, 1)$. So the total available consumption goods at t is

$$Y(d_t^1, d_t^2) = d_t^1 + d_t^2 \in \{2e_h, e_h + e_l, 2e_l\}$$

Both agents have period utility function $u(c) = \ln(c)$ for consuming c units of current period goods. Each agent maximizes his/her lifetime expected discounted utility, with common discount factor $\beta \in (0, 1)$.

1. (12 points) Define and solve the planner's problem, assuming the two agents have equal Pareto weights.
2. (24 points) Suppose that the two agents can trade date/state-contingent Arrow-Debreu securities at date-0 competitive security market.
   (a) Define a competitive equilibrium in this AD market economy.
   (b) Solve the AD competitive equilibrium, including both equilibrium allocations and prices.
   (c) How does your equilibrium outcome compare to the solution to the planner's problem in (a)? Show why your claim hold.
3. (9 points) In the AD equilibrium setting, we have assumed that only claims on future dividend are traded. Use equilibrium prices solved above to price the value of a Lucas tree at an arbitrary date $t \geq 0$. Does the value of a tree at t depend on date-t dividend realization? How? Explain.

Quarter 2

A social planner seeks to maximize the utility of the representative household,

$$\sum_{t=0}^{\infty} \beta^t (\log C_t - v(N_t)). \tag{1}$$

The planner faces a resource constraint,

$$K_{t+1} = F(K_t, A_t N_t) + (1-\delta)K_t - C_t - \phi(K_t, K_{t+1}). \tag{2}$$

The function ϕ represents an adjustment cost on investment, weakly convex in its arguments. The planner chooses a sequence for $\{C_t, N_t, K_{t+1}\}_{t=0}^{\infty}$ to solve this problem, taking as given K_0.

1. (9 points) Write down the sequence problem for the planner.
2. (9 points) In this part only, suppose that F is homogeneous of degree 1 in K_t and ϕ is homogeneous of degree 1 in (K_t, K_{t+1}). Are marginal value and the average value of capital equal?
3. (9 points) Write down the Bellman equation for the planner.
4. (9 points) Derive first order conditions and an envelope condition. Manipulate them to get two equations relating $\{C_t\}$, $\{N_t\}$, and $\{K_t\}$.
5. (9 points) Assume $A_t = (1+g)^t$ for $g > 0$. What are the minimal conditions on the function F and ϕ that you need to impose to ensure the existence of a balanced growth path, whe re $K_t = k(1+g)^t$, $C_t = c(1+g)^t$, and $N_t = n$? Find three equations relating these n, k, c using your answer from the previous part and the resource constraint.

Quarter 3

There are two infinitely-lived ex ante identical agents $i = 1, 2$ and a single nonstorable consumption good each date. Agents receive endowments of the consumption good stochastically: if one agent has endowment $1+\varepsilon$, the other agent has $1-\varepsilon$. Let $s_t \in S = \{1, 2\}$ denote the agent who has endowment $1+\varepsilon$ at date t. Assume that $\{s_t\}_{t=0}^{\infty}$ is a sequence of i.i.d. random variables with

$$Prob\{s_t = 1\} = Prob\{s_t = 2\} = \frac{1}{2}.$$

The parameter $\varepsilon \in [0, 1)$ measures the variability of the endowment process. Both agents maximize lifetime expected discounted utility

$$E \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where E is the expectation operator, $\beta \in (0, 1)$, and $u(\cdot)$ is continuous, twice continuously differentiable, $u'' < 0 < u'$ on $(0, \infty)$, and satisfies the Inada condition $u'(c) \to \infty$ as $c \to 0$.

In this environment, both agents have incentive to share their endowment risk. However, they face a commitment problem: either agent can renege on any risk-sharing agreement obligations at any time. The punishment for such an action is autarky thereafter.

1. (15 points) Let V^{fb} denote the lifetime utility of the first best (perfect risk-sharing) allocation. Let $V_{aut}(1+\varepsilon)$ and $V_{aut}(1-\varepsilon)$ denote the lifetime expected utility under autarky, for the current rich and poor agent, respectively.
   (a) What are V^{fb}, $V_{aut}(1+\varepsilon)$ and $V_{aut}(1-\varepsilon)$? Express them explicitly in term of u, β and ε.
   (b) Show that there exists a unique $\varepsilon_1 \in (0, 1)$ such that $\varepsilon_1 = \text{argmax}_{\varepsilon} V_{aut}(1+\varepsilon)$, and that $V_{aut}(1+\varepsilon_1) > V^{fb}$.
2. (10 points) Formulate the recursive problem for the optimal sustainable risk-sharing allocation such that no agent has incentive to renege at any time.
3. (20 points) Assume that the solution to the optimal sustainable risk-sharing problem define above is characterized as follows: the agent with high endowment $1+\varepsilon$ consumes $1+x(\varepsilon)$ and the agent with low endowment $1-\varepsilon$ consumes $1-x(\varepsilon)$, with $x(\varepsilon) \in [0, \varepsilon]$. That is, with optimal sustainable risk-sharing, consumption oscillates between $1+x(\varepsilon)$ and $1-x(\varepsilon)$ instead of $1+\varepsilon$ and $1-\varepsilon$ as under autarky. (No need to prove this.)
   (a) What can you say about $x(\varepsilon)$, i.e., what conditions should it satisfy? Explain.
   (b) Let ε_2 be such that $V_{aut}(1+\varepsilon_2) = V^{fb}$ and $\varepsilon_2 > \varepsilon_1$. Assume that such ε_2 exists and $\varepsilon_2 < 1$. Discuss intuitively, on each of the following interval
      (i) $\varepsilon \in [0, \varepsilon_1)$
      (ii) $\varepsilon \in [\varepsilon_1, \varepsilon_2)$
      (iii) $\varepsilon \in [\varepsilon_2, 1)$

      among full risk-sharing, partial risk-sharing, and autarky, which option is achievable with the optimal sustainable risk-sharing contract.

Quarter 4

This problem will have you work with a model with a representative household that consumes and supplies labor, a representative firm that produces final goods, a continuum of monopolistically competitive firms that produce intermediate goods, and a government that levies taxes and makes transfers. Time is discrete with an infinite horizon. There is no money in this economy, so all variables are real, not nominal.

1. (7 points) We'll start by setting up the problem facing the representative household. The household's period-t utility is:
$$c_t - v(\ell_t), \tag{3}$$
where c_t is date-t consumption and ℓ_t is date-t labor supply. The function $v(\cdot)$ captures the disutility of labor, and it is assumed to be increasing and convex. The household's labor income is $w_t \ell_t$, where w_t is the real wage. The household is also assumed to own the firms in this economy, so the firms rebate a real dividend d_t to the household. The household also gets a lump-sum transfer g_t from the government. (If $g_t < 0$, then this is understood to be a lump-sum tax.) For simplicity, assume that there is no borrowing nor saving. The date-t budget constraint is therefore:
$$c_t = w_t \ell_t + d_t + g_t. \tag{4}$$
From the household's perspective, w_t, d_t, and g_t are exogenous. Because there is no borrowing nor saving, the household's problem is static, not dynamic. Write down the household's date-t optimization problem and the associated first-order condition(s).

2. (7 points) The final consumption good in this economy is produced in a perfectly competitive market, so we can look at a single representative firm. The final-goods firm produces output y_t using a continuum of intermediate goods $\{y_{i,t} \mid i \in [0,1]\}$ with the technology:
$$y_t = \left(\int_0^1 y_{i,t}^{1-\nu} di \right)^{\frac{1}{1-\nu}}. \tag{5}$$
The final consumption good is the numeraire, so its price is normalized to one. The price of the i^{th} intermediate good $y_{i,t}$ is denoted $p_{i,t}$, which the final-goods firm takes as given. The final-goods firm is assumed to maximize profits period-by-period. Write down the profit-maximization problem. Take the first-order condition with respect to $y_{k,t}$, and use it to provide an expression for $p_{k,t}$ in terms of $y_{k,t}$, y_t, and ν. Your final answer should not contain an integral.

3. (8 points) Each intermediate firm i produces its good with technology $y_{i,t} = \ell_{i,t}$, where $\ell_{i,t}$ is the quantity of labor hired by firm i. The firm also faces adjustment costs if it wants to change the quantity of labor it hires. Specifically, it loses $\frac{\alpha}{2}(\ell_{i,t} - \ell_{i,t-1})^2$ units of output from changing $\ell_{i,t}$, where $\alpha > 0$ is a parameter that captures the severity of the adjustment cost. The firm pays its workers $w_t \ell_{i,t}$ for their time, but the firm is also subject to a proportional payroll tax $\tau_t w_t \ell_{i,t}$. The tax rate τ_t follows an exogenous Markov process that the firm takes as given. The real dividend that the firm rebates to the household at date t is:
$$d_{i,t} = p_{i,t} y_{i,t} - (1 + \tau_t) w_t \ell_{i,t} - \frac{\alpha}{2} (\ell_{i,t} - \ell_{i,t-1})^2. \tag{6}$$

The intermediate firms take w_t as given. However, firm i understands that it faces a downward-sloping demand curve for its good, and the demand for good i is characterized by your solution to question 2. (Aggregate output of final goods y_t enters into firm i's demand curve, and firm i takes y_t as given.) Firm i's objective at date t is to maximize:

$$\sum_{k=0}^{\infty} \beta^k \mathbb{E}_t \left[d_{i,t+k} \right], \tag{7}$$

subject to the constraints implied by the demand curve and the production technology.

(a) Write down the firm's Bellman equation and call it $f(\cdot)$. I recommend that you consolidate the production technology and the demand curve into equation (6) to eliminate $p_{i,t}$ and $y_{i,t}$. Although that's not strictly necessary, it allows you to write $d_{i,t}$ as a function only of $\ell_{i,t}$ and things that firm i takes as given.

(b) What are the first-order and envelope conditions for the firm? (Assume that the solution is interior, and ignore any non-negativity constraints. You don't have to show that the Bellman equation is differentiable.) Use the envelope condition to eliminate the derivative of the Bellman equation from the first-order condition.

4. (8 points) Given a initial condition for $\{\ell_{i,t} \mid i \in [0,1]\}$ and an exogenous stochastic process for τ_t, an equilibrium consists of (random) sequences of aggregate variables $\{c_t, \ell_t, w_t, d_t, y_t\}_{t=0}^{\infty}$ and variables for the intermediate-goods firms $\{p_{i,t}, \ell_{i,t}, d_{i,t}, y_{i,t} \mid i \in [0,1]\}_{t=0}^{\infty}$ such that:
- Given w_t, d_t, and g_t, c_t and ℓ_t solve the household's problem.
- Given $\{p_{i,t} \mid i \in [0,1]\}$, $\{y_{i,t} \mid i \in [0,1]\}$ solves the final-goods firm's problem.
- Given w_t and τ_t, the intermediate-firm variables solve each intermediate firm's problem.
- The final-goods market is competitive, resulting in zero profits: $y_t = \int_0^1 p_{i,t} y_{i,t} di$.
- The government's budget is balanced: $g_t = \tau_t w_t \ell_t$.
- Markets clear: $\ell_t = \int_0^1 \ell_{i,t} di$ and $d_t = \int_0^1 d_{i,t} di$.

For the remainder of the exam, assume that the equilibrium is symmetric:

$$(p_{i,t}, \ell_{i,t}, d_{i,t}, y_{i,t}) = (p_{j,t}, \ell_{j,t}, d_{j,t}, y_{j,t}), \quad \forall i, j, t. \tag{8}$$

This is equivalent to assuming that the initial condition $\ell_{i,-1}$ is the same for all firms i. Also, for the remainder of the exam, assume that the household's utility function takes the form:

$$v(\ell_t) = \frac{\phi}{2} \ell_t^2. \tag{9}$$

Define $x_t \equiv 1 + \tau_t$. (We'll provide an explicit stochastic process for x_t later.) Combine equilibrium conditions to obtain an expectational difference equation that contains only y_t, y_{t-1}, y_{t+1}, and x_t, plus model parameters.

5. (7 points) Assume that the log of x_t follows an AR(1) process:

$$\log(x_t) = \rho \log(x_{t-1}) + \epsilon_t, \quad \epsilon_t \overset{\text{i.i.d.}}{\sim} \mathrm{N}\left(0, \sigma^2\right), \tag{10}$$

and assume that this process is stationary. The above specification implies that the steady-state value of x_t is $\bar{x} = 1$. You don't have to solve for the steady-state value of y_t, but assume that it has a steady state value $\bar{y} > 0$. As in class, let "hats" over variables denote log deviations from steady state: $\hat{x}_t \equiv \log(x_t/\bar{x})$ and $\hat{y}_t \equiv \log(y_t/\bar{y})$. Log-linearize equation (10) and your answer to question 4. You should get two linear expectational difference equations that contain only $\hat{x}_t$, $\hat{y}_t$, ϵ_t, and possibly leads and lags of these variables.

6. (8 points) For the remainder of the exam, assume that $\beta = 0$, meaning that intermediate-goods firms are myopic in their decision making. You don't have to re-solve the firm's problem; you'll just set β to be zero in the difference equations you've derived. Also, for technical reasons, assume that $\alpha \neq \phi$.

   (a) Show that $\hat{y}_t$ follows an AR(p) process, where p is a finite integer. Either provide an expression for the autoregressive lag polynomial, or provide the autoregressive coefficients. Hint: Use lag polynomials.

   (b) What condition (or conditions) have to be satisfied for $\hat{y}_t$ to be stationary? Don't just state the generic condition for stationarity in an ARMA process; please state the condition(s) in terms of the coefficients that appear in the difference equation you just derived.