Fomenting Conflict*

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Abstract

I study information disclosure as a means to create conflict. A sender aims to keep two parties engaged in a war of attrition and reveals information about their relative strength. In the unique Markov Perfect Equilibrium, the sender employs "shifting rhetoric": she alternates pipetting good and bad news about each party, so that neither appears too strong. Information designed to induce one party to continue fighting weakens the other party's incentives. This spillover effect may lead to early resolution and leave the sender worse off. With commitment, the sender provides delayed noisy disclosures. A partisan sender, who favors one party, instead provides information which leads to immediate resolution.

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> The followers must feel humiliated by the ostentatious wealth and force of their enemies. [...] However, the followers must be convinced that they can overwhelm the enemies. Thus, by a continuous shifting of rhetorical focus, the enemies are at the same time too strong and too weak.
>
> — Umberto Eco[1]

1 Introduction

Fomenting conflict is a staple tactic of autocrats, politicians, and interest groups. A dictator may foster division between opposition groups to solidify his rule. An imperial power may sow distrust among rivals to weaken their resistance. An employer may create tensions between different employee groups to weaken their bargaining power. A political party may galvanize supporters by pitting them against adversaries real or imagined.[2] Manipulating information is key to such strategies. Woodward (1995), Brass (1997), and Woodward (2002), provide exhaustive accounts of how politicians distorted news reporting to foster ethic tension. Recent research points to congressional speech (Gentzkow et al. (2016)) and partisan news (Martin and Yurukoglu (2017), Durante et al. (2019)) as drivers of political polarization.

A central question in the discourse on conflict is credibility: "A major puzzle [is why] publics follow leaders down paths that seem to serve elite power interests most of all" (Fearon and Laitin (2000)) and "Why are hate-creating stories powerful even when they are false or essentially uninformative?" (Glaeser (2005)).

In this paper, I characterize how a sender manipulates information to keep two players engaged in conflict. When the sender lacks commitment, the optimal policy features "shifting rhetoric." Whenever one player appears too strong, the sender reveals information that makes him appear weaker, and whenever one player appears too weak, the sender reveals information that makes him appear stronger. Despite this apparent inconsistency in messaging, the sender's strategy is credible, and succeeds at least temporarily in prolonging the conflict. From an ex-ante perspective, however, manipulating information may be self-defeating. Disclosures aimed at preventing one player from quitting weaken the other player's incentives. This spillover effect reduces each player's value from participating and may lead to an earlier resolution. By contrast, when the sender has commitment power, she

[1] See Eco (1995).

[2] See respectively Ezrow and Frantz (2011), p. 223, Abbott (1911), p. 58 ("Rome did her best to develop the spirit of discord among [her adversaries] by arraying community against community and the aristocracy against the democracy."), Levitt et al. (1993), and Eco (1995). See Glaeser (2005) and Posner et al. (2010) for more examples.

delays noisy disclosures to nudge players to continue fighting instead of gradually releasing information. The sender's motive is key for the dynamics of information. A partisan sender, who aims to see one of the players win, has no inherent use of delaying the conflict. Then, conflict ends instantaneously, both when the sender has commitment and when she has none.

I model conflict as a continuous-time war of attrition in which the players' relative strength is uncertain. Two players (he) pay a flow cost each instant and choose when to quit. When both players continue, the game ends exogenously with a constant Poisson rate. A binary state v, which is unknown to the players, determines whether player 1 ($v = 1$) or player 2 ($v = 0$) wins at that time. Thus, player 1 is strong when $v = 1$ and player 2 is strong when $v = 0$. The belief that $v = 1$ parameterizes the perceived strength of player 1.[3]

A sender (she) aims to prolong the war.[4] She receives a positive flow payoff when both players continue and zero when the game ends. The sender knows v and designs dynamic disclosure strategies, which are publicly observed by both players. A disclosure strategy consists of a cumulative probability of verifiably disclosing that $v = 1$ and another cumulative probability of disclosing that $v = 0$. The sender has full flexibility in adjusting these probabilities over time. Additionally, players learn about v via exogenous news, which takes the form of an arithmetic Brownian Motion. Thus, the sender controls only part of the information players receive.[5]

I first characterize the unique Markov Perfect Equilibrium (MPE), in which the sender cannot commit to disclosures ahead of time. This equilibrium is characterized by two thresholds of beliefs. At the upper threshold, player 2 quits, since it is likely player 1 is strong. At the lower threshold, player 1 quits instead. In response to these strategies, the sender pipets different kinds of information. At the upper threshold, she gradually reveals whether $v = 1$, i.e. whether player 1 is truly strong, while at the lower threshold, she gradually reveals whether $v = 0$, i.e. whether player 2 is truly weak. Once this information is revealed, the game ends. However, without disclosure, the belief reflects away from either threshold,

[3] This modeling choice conveniently allows for uncertainty about each player's relative strength without introducing private information or multidimensional beliefs. Wars of attrition in which players have private information have been studied by Cetemen and Margaria (2021) and are not the focus of this paper.

[4] In the economics literature, *why* a politician creates conflict has been studied in Acemoglu et al. (2004), Glaeser (2005), and Padró i Miquel (2007). Instead, this paper focuses on *how* a politician creates conflict. In the political science literature, there is broad consensus that fostering conflict is intentional and benefits those fostering it. See e.g. Fearon and Laitin (2000), who focusing on ethnic conflict note that "If there is a dominant or most common narrative in the texts under review, it is that large-scale ethnic violence is provoked by elites seeking to gain, maintain, or increase their hold on political power." Following this consensus, I take the sender's motive as given. In Section 4.4, I study partisan persuasion, where the sender benefits when one particular player wins.

[5] Introducing exogenous information is necessary to generate dynamics in the sender's optimal policy. Otherwise, the sender provides a one-shot disclosure and the game either ends immediately or continues according to a standard war of attrition. See Zhang and Zhou (2016) for a characterization of this case.

inducing both players to continue. Thus, when the exogenous information makes either player appear too strong, the sender provides disclosures to make that player appear weaker. Over time, she alternates disclosures that make player 1 appear strong with disclosures that make player 1 appear weak. Hence, she employs "shifting rhetoric," which keeps the players engaged in conflict.

While this strategy is the sender's best response to players' stopping decisions, it is ex-ante suboptimal. This is due to a spillover effect: information aimed to prevent one player from quitting hurts the other player's incentives and vice versa. This renders both players more willing to quit. To see this explicitly, consider the no-information benchmark. When the belief is sufficiently high, player 2 is convinced that $v = 1$ and her value from continuing is relatively low. At some threshold belief, player 2 quits and player 1 wins. Suppose that the sender pipets information at that threshold to prolong the game. Then, with some probability, the belief reflects downwards and both players continue. But this reduces the value of player 1 at the threshold, since he no longer wins with certainty. Anticipating the sender's disclosures, player 1 has a lower value from continuing at all other beliefs. The analog holds for player 2, which suggests that the game may end earlier in equilibrium. In line with this intuition, I show that the sender's optimal policy in the MPE is actually the *minmax* policy for both players. Thus, while the sender's pipetting strategy is optimal at each instant, it minimizes each player's value from continuing. As a result, the game may end earlier on average in unique MPE, compared to the no-information benchmark.

With commitment, the sender can improve. In particular, she can delay noisy but valuable information to induce each player to continue. When the sender does disclose information, she must balance both players' incentives. Disclosure that favors player 1 may be detrimental for player 2 and vice versa. I show that the optimal disclosure maximizes each player's value when that player is about to quit and conversely minimizes that player's value when that player is about to win. Intuitively, promising higher value when a player is about to quit has a higher impact on that player's stopping decision than a higher value when that player is close to winning. Thus, the sender maximizes the former and minimizes the latter.

The sender's motives crucially affect the dynamics of information disclosure. When the sender benefits from, say, player 1 winning, the unique MPE features immediate full disclosure. Intuitively, for any stopping strategy the players, delaying information is not valuable for the sender and her best response is to instantly disclose information to maximize the likelihood that player 1 wins. The unique MPE features full disclosure because of the sender's lack of commitment. To see this, suppose that player 2 quits at some threshold. Then, whenever the belief is below that threshold, it is optimal for the sender to instantly disclose information to bring the belief back to the threshold. But given this strategy, player 2 has a profitable deviation. He can wait for a small amount of time, and if exogenous information causes the belief to go downwards, the sender provides information which may lead player 2 to win. To render this deviation unprofitable, the belief at the threshold must be degenerate. Thus, in the unique MPE, the sender provides full disclosure. As in the baseline model, the sender can improve if she has commitment power. As in the case without commitment, the sender does not benefit from delay and instantaneously provides information. Specifically, the sender provides instant noisy disclosure, to induce a sufficiently high belief, and then threatens player 2 with the minmax strategy to induce him to quit. This strategy maximizes the instantaneous probability that player 1 wins, and dominates any dynamic strategy.

Technical Contribution. In constructing the MPE, the paper makes several technical contributions. To characterize the equilibrium, I solve for a fixed point between the sender's disclosure strategy and both players' stopping strategies. I first formulate a general verification theorem for stopping problems, given a general class of dynamic disclosure strategies for the sender (Proposition 7). This theorem may be of use to other work, and, to my knowledge, no such result exists in the mathematics or economics literature. Then, I show how to convert each player's problem into a standard stopping problem with a terminal payoff. Solving this problem is involved, since the arrival of exogenous news precludes using closed-form solutions.[6] I use the characterization of excessive function in Dayanik and Karatzas (2003),[7] which allows me to transform the state space and characterize tangency properties of certain functions which then correspond to the solution of the optimal stopping problem. Then, I show that by varying the terminal payoff, the solution to the standard stopping problem can be made to equal the solution given pipetting. Given players' strategies, I then formulate another verification theorem for the sender (Proposition 9), which may also be of independent interest, and I use this verification theorem to characterize the sender's best response. Overall, I expect the approach in this paper to generalize to games involving dynamic disclosure and optimal stopping.

[6] See e.g. Equations (7) and (11) and note that the volatility terms are nonlinear. In related work (see the literature review) Orlov et al. (2020) characterize an MPE between a sender and one receiver without exogenous news. Their construction relies on closed form solutions for the receiver's optimal stopping problem, which cannot be obtained in my setting.

[7] See also De Angelis et al. (2018).

2 Literature

This paper contributes to the literature on dynamic persuasion.[8] The closest papers are Orlov et al. (2020), Bizzotto et al. (2021), and Ely et al. (2021). Orlov et al. (2020) consider pipetting strategies with a single receiver, for whom the value of exercising an option is affected by an exogenous process and the sender's disclosures. This paper features crucial differences in setup and results: (1) The sender faces two receivers and disclosures simultaneously affect the incentives of both. As a result, the sender uses qualitatively different strategies, depending on which player is about to quit. (2) Players observe exogenous news, so that the sender only has imperfect control over the players' information. (3) In the unique MPE, the sender may be worse off compared to disclosing no information, whereas in Orlov et al. (2020) the sender is always better off (see their Prop. 2). This is due to "spillover" effects from disclosure, i.e. when the sender prevents player 2 from quitting, she hurts player 1's incentives, which are absent from Orlov et al. (2020). (4) In Orlov et al. (2020), the optimal commitment solution features delayed full disclosure. In this paper, delayed full disclosure is suboptimal, since the sender must balance each both players' incentives. This renders delayed noisy disclosures optimal.

Ely et al. (2021) study a dynamic contest in which multiple players exert effort and generate breakthroughs. These breakthroughs are observed only by the principal, who then chooses how to reveal them publicly to maximize contestants' effort. The optimal policy features a review cycle, which appraises players about others' successes at fixed intervals. In Ely et al. (2021), the sender controls all information and can commit to dynamic policies. By contrast, in this paper, players observe exogenous news and the sender lacks commitment over time, which generates qualitatively different predictions.

Bizzotto et al. (2021) consider a sender who lacks commitment and aims to persuade a single receiver who observes exogenous information. In equilibrium, the sender induces inefficient delay in the receiver's decision in order to increase the likelihood that the receiver takes a favorable action. In this paper, the sender instead benefits from delay, but compared to the commitment solution, the game ends too early. In this sense, this paper generates opposite predictions to Bizzotto et al. (2021).

Also related are Basak and Zhou (2020a) and Basak and Zhou (2020b), who study dynamic persuasion in regime change settings, where the sender faces a continuum of heterogeneously informed receivers. In both papers, agents choose when to attack without observing other agents decisions. In Basak and Zhou (2020a), players receive static private informa-

[8] See Au (2015), Ely et al. (2015), Ely (2017), Renault et al. (2017), Ball (2019), Che and Mierendorff (2019), Kolb and Madsen (2019), Ely and Szydlowski (2020), Zhao et al. (2020), Che et al. (2020), and Smolin (2021).

tion, while in Basak and Zhou (2020b), players information is correlated and arrives at fixed intervals. Their optimal policies feature periodic viability tests and disaster alerts, which differ qualitatively from the pipetting strategies in this paper.

A large literature starting with Smith (1974) studies wars of attrition. Closest to this paper is the literature on wars of attrition with stochastically evolving payoffs (Gieczewski (2020), Georgiadis et al. (2022)) or exogenous uncertainty and learning (Murto (2004), Kim and Lee (2014), and Cetemen and Margaria (2021)). Gul and Pesendorfer (2012) consider a war of attrition in which two players pay a cost and a decision maker observes exogenous information unless one player quits. Meyer-ter Vehn et al. (2018) study a game where waiting is costly and two players make sequential proposals which reveal their private information. Relative to these papers, my focus is on characterizing the disclosure policy of a sender who wants to prolong the war of attrition.

Finally, this paper is related to literatures studying political persuasion (e.g. Alonso and Câmara (2016), Chan et al. (2019), and Boleslavsky et al. (2021)) and polarization (e.g. Martin and Yurukoglu (2017), Durante et al. (2019), and Ottinger and Winkler (2022)).

3 Model

Sender and Receivers. Two players $i = 1, 2$ are engaged in a continuous-time war of attrition. Each player pays a flow cost $c > 0$ and chooses when to stop. When player i stops, he receives 0, player $-i$ receives 1,[9] and the game ends. When both players continue, the game ends exogenously at time τ_E, where τ_E is the first arrival time of a Poisson process with rate λ. At time τ_E, player 1 receives v, while player 2 receives $1 - v$. The value $v \in \{0, 1\}$ is unknown to both players, and their common prior belief is given by $p_{0-} \equiv \Pr(v = 1) \in (0, 1)$. Intuitively, if $v = 1$, player 1 wins at time τ_E, while if $v = 0$, player 2 wins. The belief p_{0-} parameterizes the ex-ante strength of player 1. A higher belief increases player 1's value of continuing and decreases player 2's value of continuing. A sender knows v and designs a disclosure policy which is publicly observed by the players. She wishes to prolong the war, and receives a flow payoff of $w > 0$ as long as both players continue and a payoff of zero when the game ends.[10]

Exogenous Information. Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the usual conditions with filtration $\mathbb{F} \equiv \{\mathcal{F}_t\}_{t\geq 0}$, on which there is a Brownian Motion $\{B_t\}_{t\geq 0}$ and the Poisson process described above. The Brownian Motion and the Poisson process are independent

[9] Throughout the paper, $-i$ denotes the player other than player i.

[10] Section 4.4 considers the case when the sender wishes to see player 1 win.

from each other. Public information about v arrives via the arithmetic Brownian Motion

$$dX_t = vdt + \sigma dB_t, \tag{1}$$

with $X_0 = 0$. I denote with $\mathbb{F}^X = \{\mathcal{F}_t^X\}_{t\geq 0} \subset \mathbb{F}$ the filtration generated by $\{X_t\}_{t\geq 0}$. Absent any information from the sender, player 1 and 2's common belief $p_t \equiv \Pr(v = 1|\mathcal{F}_t^X)$ follows[11]

$$dp_t = \sigma\left(p_t\right) d\hat{B}_t, \tag{2}$$

where $\{\hat{B}_t\}_{t\geq 0}$ is a Brownian Motion on $\mathbb{F}$, and where I define with slight abuse of notation[12]

$$\sigma\left(p\right) = \frac{p\left(1-p\right)}{\sigma}.$$

Disclosure Policies. The sender chooses a disclosure policy $D \equiv \{D_t^L, D_t^R\}_{t\geq 0-} \in \mathcal{D}$ defined as follows. For each time t, D_t^L is the cumulative probability of verifiably disclosing that $v = 0$ up to time t, and D_t^R is the cumulative probability of verifiably disclosing that $v = 1$ up to time t.[13] Here, the set of admissible disclosure policies $\mathcal{D}$ is the set of $\mathbb{F}$-adapted, right-continuous-left-limit, finite variation processes with weakly increasing components, such that $D_{0-}^L = D_{0-}^R = 0$ and $\{D_t^L, D_t^R\} \in [0,1]^2$ for all $t \geq 0-$.[14] To save notation, I write $D_t \equiv \{D_t^L, D_t^R\}$. I also denote with $D^R = 0$ the strategy which never discloses that $v = 1$, i.e. $D_t^R = 0$ for all $t \geq 0-$, and similarly denote with $D^L = 0$ the strategy which never discloses that $v = 0$, i.e. $D_t^L = 0$ for all $t \geq 0-$.

A particular subset of disclosure strategies are *pipetting* strategies. For a given threshold $\bar{p} \in (0,1)$ define a *right-pipetting* strategy $D^R(\bar{p}) \equiv \{D_t^R\}_{t\geq 0-}$ as follows. If $p_{0-} \geq \bar{p}$, then

[11] See Liptser and Shiryaev (2001), Th. 9.1, p. 355, and Ex. 1, p. 371.

[12] For every $p \in (0,1)$, there exists an $\varepsilon > 0$ such that

$$\int_{p-\varepsilon}^{p+\varepsilon} \frac{1}{\sigma\left(s\right)^2} ds < \infty.$$

Hence, the belief process is well-defined on the interval $(0,1)$.

[13] More precisely, fix a message space $M = \{0,1\}$. D_t^R is the cumulative probability that the sender sends message $m_t = 1$ at time t conditional on $v = 1$. Similarly, D_t^L is the cumulative probability that the sender sends message $m_t = 0$ at time t conditional on $v = 0$. Thus, $p_{0-}(1 - D_t^R) + (1 - p_{0-})(1 - D_t^L)$ is the ex-ante probability that the sender sends no message before time t.

[14] This definition is less general than the one used in Orlov et al. (2020). In their Lemma 1, Orlov, Skrzypacz, and Zryumov argue that any dynamic persuasion policy is equivalent to a disclosure policy whenever the stopping set is an interval of the form $[a,1]$. However, the stopping set in Proposition 2 does not take this form, so Lemma 1 from Orlov et al. (2020) does not apply to the current setting. As in Orlov et al. (2020), the sender can disclose information instantaneously before any exogenous uncertainty realizes. As in that paper, $t = 0-$ denotes the time immediately preceding $t = 0$, so that $dD_0^L = D_0^L - D_{0-}^L$ and $dD_0^R = D_0^R - D_{0-}^R$ describe information revealed before any exogenous uncertainty is observed.

D_0^R satisfies[15]

$$\bar{p} = \frac{p_{0-}\left(1 - D_0^R\right)}{p_{0-}\left(1 - D_0^R\right) + 1 - p_{0-}}. \tag{3}$$

If $p_{0-} < \bar{p}$, then $D_0^R = 0$. For all $t \geq 0$,

$$dD_t^R = \left(1 - D_{t-}^R\right) d\mathbb{1}\{p_t^{max} \geq \bar{p}\}, \tag{4}$$

where $p_t^{max} = \sup_{s \leq t} p_t$.[16] Similarly, for a given threshold $\underline{p} \in (0,1)$ with $\underline{p} < \bar{p}$, define a *left-pipetting* strategy $D^L(\underline{p}) \equiv \{D_t^L\}_{t \geq 0-}$ as follows. If $p_{0-} \leq \underline{p}$, then D_0^L satisfies

$$\underline{p} = \frac{p_{0-}}{p_{0-} + \left(1 - p_{0-}\right)\left(1 - D_0^L\right)} \tag{5}$$

and if $p_{0-} > \underline{p}$, then $D_0^L = 0$. For all $t \geq 0$,

$$dD_t^L = \left(1 - D_{t-}^L\right) d\mathbb{1}\{p_t^{min} \leq \underline{p}\}, \tag{6}$$

where $p_t^{min} = \inf_{s \leq t} p_t$. Intuitively, at the right boundary $\bar{p}$, D_t^R either reflects the belief p_t downwards or reveals that $v = 1$, while at the left boundary $\underline{p}$, D_t^L either reflects the belief upwards or reveals that $v = 0$. Given $\{D^L(\underline{p}), D^R(\bar{p})\}$ and $p_{0-} \in (\underline{p}, \bar{p})$, the belief p_t stays within the interval $[\underline{p}, p]$ until $v = 1$ or $v = 0$ is disclosed.[17] If $p_{0-} > \bar{p}$, then D_0^R features an initial jump, so that the posterior is either $p_0 = 1$ (i.e. $v = 1$ is disclosed) or $p_0 = \bar{p}$. Similarly, if $p_{0-} < \underline{p}$, then D_0^L features an initial jump so that $p_0 = 0$ (i.e. $v = 0$ is disclosed) or $p_0 = \underline{p}$.

Beliefs. To formalize belief updating given disclosure strategy $D \in \mathcal{D}$, define two exponential random variables, ξ and η, each with parameter 1, which are independent from $\mathbb{F}$ and from each other. Given D, the belief at time t is given as

$$p_t = \begin{cases} 1 & \text{if} \quad \xi \leq -log\left(1 - D_t^R\right) \\ p_t^{ND} & \text{if} \quad \xi > -log\left(1 - D_t^R\right) \text{ and } \eta > -log\left(1 - D_t^L\right) \\ 0 & \text{if} \quad \eta \leq -log\left(1 - D_t^L\right) \end{cases}$$

[15] Note that Equation (3) is Bayes' rule, saying that conditional on no disclosure the posterior is $\bar{p}$, given the prior p_{0-}. The analog holds for Equation (5) below.

[16] This construction is analogous to the construction of reflected Brownian motion. See e.g. Harrison (2013), p. 15. Note that the right-pipetting strategy D_t^R is a continuous process for all $t > 0$, which implies that $D_{t-}^R = D_t^R$ and $p_{t-} = p_t$, unless $v = 1$ is revealed. The left-pipetting strategy D_t^L is also a continuous process for $t > 0$.

[17] See also Figure 3.

where[18]

$$dp_t^{ND} = p_{t-}^{ND}\left(1-p_{t-}^{ND}\right)\left(\frac{dD_t^L}{1-D_{t-}^L} - \frac{dD_t^R}{1-D_{t-}^R}\right) + \sigma\left(p_t^{ND}\right)d\hat{B}_t \tag{7}$$

is the belief when the sender neither conclusively reveals that $v = 1$ nor that $v = 0$ up to time t, conditional on observing $\{X_s\}_{s<t}$. Define the stopping times

$$\tau_\xi = \inf\left\{t \geq 0 : \xi \leq -log\left(1-D_t^R\right)\right\} \text{ and } \tau_\eta = \inf\left\{t \geq 0 : \eta \leq -log\left(1-D_t^L\right)\right\}, \tag{8}$$

and note that $p_t = 1$ for all $t \geq \tau_\xi$ if $\tau_\xi < \tau_\eta$, i.e. if $v = 1$ is disclosed before $v = 0$, and $p_t = 0$ for all $t \geq \tau_\eta$ if $\tau_\eta < \tau_\xi$, i.e. if $v = 0$ is disclosed before $v = 1$. We have

$$\Pr\left(\tau_\xi \leq t|\mathcal{F}_t, v = 1\right) = D_t^R \text{ and } \Pr\left(\tau_\xi \leq t|\mathcal{F}_t, v = 0\right) = 0$$

and

$$\Pr\left(\tau_\eta \leq t|\mathcal{F}_t, v = 1\right) = 0 \text{ and } \Pr\left(\tau_\xi \leq t|\mathcal{F}_t, v = 0\right) = D_t^L,$$

i.e., conditional on the history up to time t, $v = 1$ is indeed disclosed with probability D_t^R and $v = 0$ is disclosed with probability D_t^L.

Payoffs and Equilibrium. Let $\hat{\mathbb{F}}$ denote the enlarged filtration which includes the stopping times τ_ξ and τ_η. Given strategy D, players 1 and 2 choose stopping times $\tau_i \in \mathcal{T}$, where $\mathcal{T}$ is the set of $\hat{\mathbb{F}}$-adapted stopping times. Player i's stopping problem given $\tau_{-i} \in \mathcal{T}$ and the disclosure policy D is given by[19]

$$\begin{aligned} V_i\left(p_{0-}\right) &= \sup_{\tau_i \in \mathcal{T}} E\Bigg[-\int_0^{\tau_i \wedge \tau_{-i} \wedge \tau_E} e^{-rt} c dt + e^{-r\tau_{-i}} \mathbb{1}\{\tau_{-i} < \tau_i \wedge \tau_E\} \\ &\quad + e^{-r\tau_E}((2-i)p_{\tau_E} - (1-i)(1-p_{\tau_E}))\mathbb{1}\{\tau_E < \tau_i \wedge \tau_{-i}\}\Bigg] \end{aligned}$$

or equivalently

$$V_i\left(p_{0-}\right) = \sup_{\tau_i \in \mathcal{T}} E\left[\int_0^{\tau_i \wedge \tau_{-i}} e^{-(r+\lambda)t} u_i\left(p_t\right) dt + e^{-(r+\lambda)\tau_{-i}} \mathbb{1}\{\tau_{-i} < \tau_i\}\right], \tag{9}$$

[18] Here, note that the Brownian motion has continuous increments, whereas D_t^L and D_t^R may have singular increments. Thus, the terms multiplying dD_t^L and dD_t^R involve p_{t-}^{ND}, whereas the term multiplying $d\hat{B}_t$ involves p_t^{ND}.

[19] Here, $a \wedge b = \min\{a, b\}$. Note that the expectation $E[.]$ implicitly depends on the disclosure policy D. If both players stop at the same time, i.e. $\tau_i = \tau_{-i}$, I assume that both get zero. In all cases considered in the paper, the players' stopping regions are disjoint. Thus, the choice of tie-breaking rule does not affect the results.

where each player's expected flow payoff is given by

$$u_1(p_t) = \lambda p_t - c \text{ and } u_2(p_t) = \lambda(1 - p_t) - c.$$

The sender's problem given the stopping times τ_1 and τ_2 is given by

$$W(p_{0-}) = \sup_{D \in \mathcal{D}} E\left[\int_0^{\tau_1 \wedge \tau_2 \wedge \tau_E} e^{-rt} w dt\right]$$

or equivalently

$$W(p_{0-}) = \sup_{D \in \mathcal{D}} E\left[\int_0^{\tau_1 \wedge \tau_2} e^{-(r+\lambda)t} w dt\right]. \tag{10}$$

A Nash equilibrium consists of a disclosure strategy D and a pair of stopping times τ_1 and τ_2, so that D solves problem (10) given τ_1 and τ_2, and for each $i = 1, 2$, τ_i solves problem (9) given D and τ_{-i}. A disclosure policy D is *Markovian* if the induced process $\{p_t\}_{t \geq 0-}$ is a Markov process. Pipetting strategies are Markovian. A stopping strategy τ_i is Markovian if there exists a Borel set $S_i \subset [0, 1]$ such that $\tau_i = \inf\{t \geq 0 : p_t \in S_i\}$ and such that the set $[0, 1] \setminus S_i$ contains the closure of its interior.[20] A Markov Perfect Equilibrium (MPE) is a Nash equilibrium in which D, τ_1, and τ_2 are Markovian.

Parametric Assumptions. Throughout the paper, I assume that

$$\lambda > 2c,$$

i.e. the "social value" of the game for players 1 and 2 is positive, since $u_1(p) + u_2(p) = \lambda - 2c > 0$. In standard wars of attrition, the social value is assumed to be negative (i.e. $\lambda \leq 2c$), and players randomize between continuing and stopping at each instant. With exogenous information arrival, no such equilibrium exists, which I show in Appendix B.1.

4 Analysis

4.1 No-Information Benchmark

Without disclosure, player 1 continues as long as the belief p_t is sufficiently high. Specifically, for $p_t > c/\lambda$, player 1's expected flow payoff $u_1(p_t)$ is strictly positive, so continuing is a dominant strategy. After sufficiently bad news arrives, however, player 1's expected flow

[20] The latter assumption is to ensure that stopping times are well-defined. See Øksendal (2003), Ch. 2.1, p. 27.

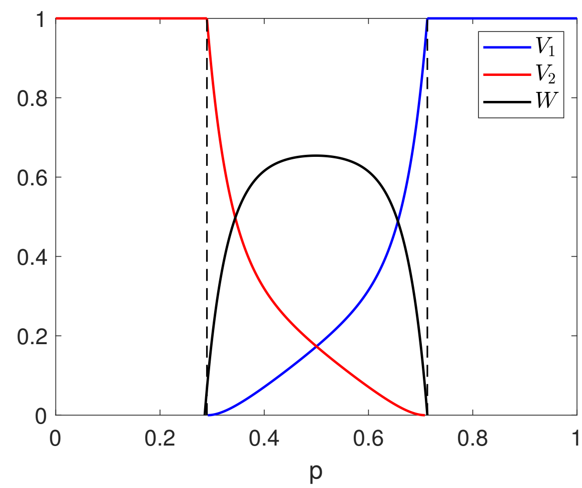

Figure 1: No-Information Benchmark. Parameters: $\lambda = 1.5$, $r = 1.5$, $c = 0.5$, $\sigma = 3$, $w = 2$.

payoff becomes negative. When the belief is sufficiently low, player 1 quits. The opposite holds for player 2, who continues whenever the belief is low and quits when the belief is high. Proposition 1 below shows that this simple intuition is correct. Figure 1 illustrates the equilibrium. All proofs are deferred to the Appendix.

Proposition 1. *Without disclosure, there exists a unique Nash equilibrium for players 1 and 2. In equilibrium, Player 1 quits whenever $p_t \leq \underline{p}_{ni}$ and Player 2 quits whenever $p_t \geq \bar{p}_{ni}$, where $0 < \underline{p}_{ni} < \frac{c}{\lambda} < 1 - \frac{c}{\lambda} < \bar{p}_{ni} < 1$. Player 1 and 2's value functions $V_{1,ni}(p)$ and $V_{2,ni}(p)$ are continuously differentiable for all $p \in [0,1]$, twice continuously differentiable for $p \notin \{\underline{p}_{ni}, \bar{p}_{ni}\}$, and for $p \in (\underline{p}_{ni}, \bar{p}_{ni})$ are the unique solutions to the ODEs*

$$(r + \lambda) V_{1,ni}(p) = \lambda p - c + \frac{1}{2}\sigma(p)^2 V''_{1,ni}(p) \tag{11}$$

with boundary conditions $V_{1,ni}(\underline{p}_{ni}) = V'_{1,ni}(\underline{p}_{ni}) = 0$ and $V_{1,ni}(\bar{p}_{ni}) = 1$, and

$$(r + \lambda) V_{2,ni}(p) = \lambda(1 - p) - c + \frac{1}{2}\sigma(p)^2 V''_{2,ni}(p) \tag{12}$$

with boundary conditions $V_{2,ni}(\underline{p}_{ni}) = 1$ and $V_{2,ni}(\bar{p}_{ni}) = V'_{2,ni}(\bar{p}_{ni}) = 0$. For $p \in (\underline{p}_{ni}, \bar{p}_{ni})$, $V_{1,ni}(p)$ is strictly increasing and strictly convex, while $V_{2,ni}(p)$ is strictly decreasing and strictly convex. For $p > \bar{p}_{ni}$, $V_{1,ni}(p) = 1$ and $V_{2,ni}(p) = 0$, and for $p < \underline{p}_{ni}$, $V_{1,ni}(p) = 0$ and $V_{2,ni}(p) = 1$. The sender's value function $W_{ni}(p)$ is continuously differentiable for $p \in [0,1]$, twice continuously differentiable for $p \notin \{\underline{p}_{ni}, \bar{p}_{ni}\}$, and is the unique solution to the ODE

$$(r + \lambda) W_{ni}(p) = w + \frac{1}{2}\sigma(p)^2 W''_{ni}(p) \tag{13}$$

with boundary conditions $W_{ni}(\underline{p}_{ni}) = W_{ni}(\bar{p}_{ni}) = 0$*. For* $p \in (\underline{p}_{ni}, \bar{p}_{ni})$*,* $W(p)$ *is strictly concave and for* $p \notin (\underline{p}_{ni}, \bar{p}_{ni})$*,* $W(p) = 0$*.*

To prove this result, I construct a fixed point between player 1 and 2's stopping thresholds. That is, assuming that player 2 stops at some threshold $\bar{p}$, I solve player 1's optimal stopping problem (Equation (32)). Since players HJB equations are nonlinear in p, I cannot use closed form solutions and use the characterization of excessive functions in Dayanik and Karatzas (2003). After suitably transforming the state space, player 1's optimal stopping threshold is determined by constructing a concave majorant for the transformed stopping payoff (Equation (42)). The analog approach characterizes player 2's stopping threshold assuming that player 1 stops at some $\underline{p}$. In the transformed space, the stopping threshold of player i is bounded below a function given the threshold for player $-i$ (Lemma 6), which allows me to show that a unique fixed point in threshold strategies exists. Then, I show that any equilibrium must be in threshold strategies.

4.2 Markov Perfect Equilibrium

In the unique MPE, the sender pipets information that makes player 1 appear weaker whenever player 2 is about to quit. Similarly, she pipets information that makes player 1 appear stronger whenever player 1 is about to quit. Thus, the sender uses disclosures to ensure that no player appears too strong or too weak.

Proposition 2. *There exists a unique MPE. The unique MPE features two thresholds* $\underline{p} < \bar{p}$*, so that the sender uses the pipetting strategies* $\{D^L(\underline{p}), D^R(\bar{p})\}$ *and players 1 and 2 use the stopping strategies* $\tau_1 = \inf\{t \geq 0 : p_t < \underline{p}\}$ *and* $\tau_2 = \inf\{t \geq 0 : p_t > \bar{p}\}$*.*[21] *For* $p \in (\underline{p}, \bar{p})$*, player 1's value function* $V_1(p)$ *satisfies Equation (11) with boundary conditions* $V_1(\underline{p}) = V_1'(\underline{p}) = 0$ *and*

$$V_1'(\bar{p}) = \frac{1 - V_1(\bar{p})}{1 - \bar{p}}, \tag{14}$$

player 2's value function $V_2(p)$ *satisfies Equation (12) with boundary conditions*

$$V_2'(\underline{p}) = \frac{1}{\underline{p}}\left(1 - V_2(\underline{p})\right) \tag{15}$$

and $V_2(\bar{p}) = V_2'(\bar{p}) = 0$*, and the sender's value function* $W(p)$ *satisfies Equation (13) with*

[21] These stopping times are well defined, since the continuation region for each player, i.e. $[\underline{p}, 1]$ for player 1 and $[0, \bar{p}]$ for player 2, contains the closure of its interior. See e.g. Øksendal and Sulem (2005), Ch. 2.1, p 27.

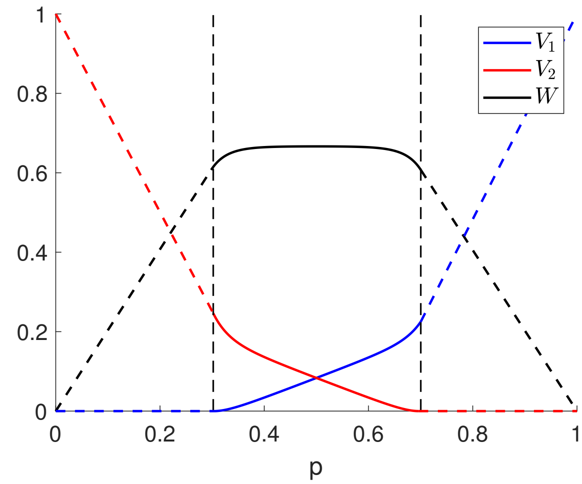

Figure 2: Markov Perfect Equilibrium. The solid lines are the value functions of players 1 and 2 and the sender on the region $[\underline{p}, \bar{p}]$. For $p > \bar{p}$, the dashed lines indicate the values when the sender randomizes between beliefs $\bar{p}$ and 1. For $p < \underline{p}$, the dashed lines indicate the values when the sender randomizes between 0 and $\underline{p}$. Parameters: $\lambda = 1.5$, $r = 1.5$, $c = 0.5$, $\sigma = 3$, $w = 2$.

boundary conditions

$$W'\left(\underline{p}\right) = \frac{1}{\underline{p}} W\left(\underline{p}\right) \tag{16}$$

and

$$W'\left(\bar{p}\right) = -\frac{W\left(\bar{p}\right)}{1-\bar{p}}. \tag{17}$$

If $p_{0-} \leq \underline{p}$, then the value functions are given by

$$\begin{aligned}
V_1\left(p_{0-}\right) &= \frac{p_{0-}}{\underline{p}} V_1\left(\underline{p}\right) \\
V_2\left(p_{0-}\right) &= \frac{p_{0-}}{\underline{p}} V_2\left(\underline{p}\right) + \frac{\bar{p} - p_{0-}}{\underline{p}} \\
W\left(p_{0-}\right) &= \frac{p_{0-}}{\underline{p}} W\left(\underline{p}\right),
\end{aligned}$$

while if $p_{0-} \geq \bar{p}$, the value functions are given by

$$\begin{aligned}
V_1\left(p_{0-}\right) &= \frac{1-p_{0-}}{1-\bar{p}} V_1\left(\bar{p}\right) + \frac{p_{0-} - \bar{p}}{1-\bar{p}} \\
V_2\left(p_{0-}\right) &= \frac{1-p_{0-}}{1-\bar{p}} V_2\left(\bar{p}\right) \\
W\left(p_{0-}\right) &= \frac{1-p_{0-}}{1-\bar{p}} W\left(\bar{p}\right).
\end{aligned}$$

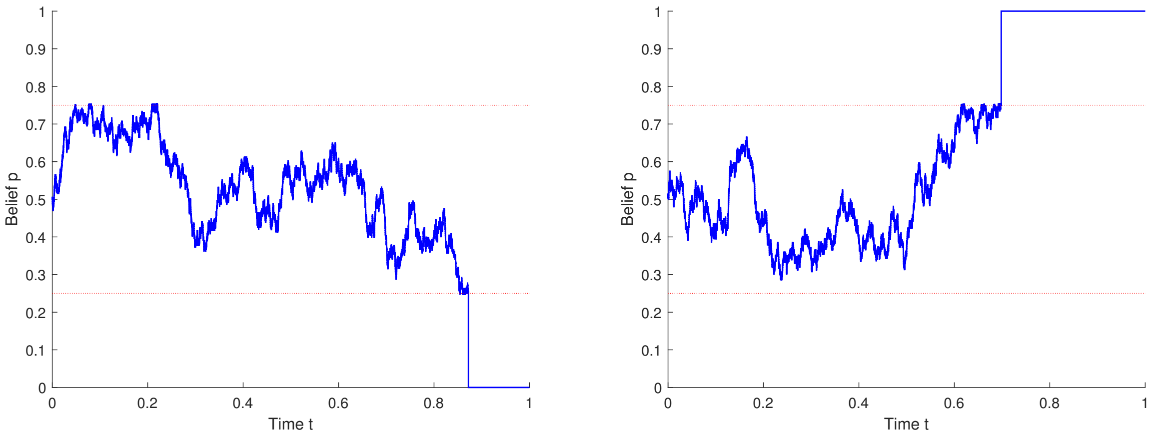

Figure 3: Simulated belief Dynamics in the MPE. Starting at $p_{0-} \in (\underline{p}, \bar{p})$, when the belief reaches $\bar{p}$, it is either reflected back inside the interval or the sender discloses that $v = 1$. The analog holds for $\underline{p}$. Parameters: $\sigma = 0.5$, $p_{0-} = 0.5$, $\underline{p} = 0.25$ and $\bar{p} = 0.75$.

Figure 2 illustrates the value functions in equilibrium, and Figure 2 illustrates the belief dynamics starting at $p_{0-} \in (\underline{p}, \bar{p})$. The method of proof, which relies on reducing players' problems to standard stopping problems, transforming the state space to use excessive functions, and using novel verification results, is described heuristically in the introduction. The following features of the equilibrium are noteworthy.

Shifting Rhetoric. Similar to the no-information benchmark, player 1 quits at any belief $p_t < \underline{p}$ and player 2 quits at any belief $p_t > \bar{p}$. Whenever $p_{t-} \in (\underline{p}, \bar{p})$, the sender provides no disclosure, i.e. $dD_t^R = dD_t^L = 0$, since both players are willing to continue. Once the belief reaches $\bar{p}$, the sender pipets information to keep the belief inside the region $[\underline{p}, \bar{p}]$, to prevent player 2 from quitting. That is, either the sender reveals that $v = 1$ or the belief reflects back inside the interval $[\underline{p}, \bar{p}]$. Similarly, once the belief reaches $\underline{p}$, either the sender reveals that $v = 0$ or the belief reflects back inside $[\underline{p}, \bar{p}]$. In this sense, the sender "shifts rhetoric" to ensure that neither player appears too strong or too weak. The boundary conditions (14)–(17) are Robin boundary conditions and arise because the players anticipate pipetting at the thresholds $\underline{p}$ and $\bar{p}$.[22]

If $p_{0-} < \underline{p}$, the sender provides instantaneous information. She chooses $D_0^L > 0$ to induce posterior beliefs $p_0 = 0$ or $p_0 = \underline{p}$ (see Equation (5)). Thus, the sender either instantaneously

[22]Importantly, I do not *impose* these boundary conditions. Instead, I show that the respective value functions must satisfy these conditions as part of the proof. See Propositions 7 and 9 in the Appendix. See e.g. Harrison (2013), p. 161, for similar conditions in the case of sticky reflection. Note that the Robin boundary conditions are equivalent to smooth pasting conditions between $V_i(p)$ and $W(p)$ and the respective values to the left of $\underline{p}$ and the right of $\bar{p}$.

reveals that $v = 0$ or moves the belief inside the region $[\underline{p}, \bar{p}]$. Similarly, if $p_{0-} \geq \bar{p}$, the sender chooses $D_0^R > 0$ to induce posteriors $p_0 = \bar{p}$ or $p_0 = 1$ (see Equation (3)).

Minmax Strategy. The pipetting strategies in Proposition 2 are the sender's best response to the stopping decisions of both players. However, a player anticipating these strategies has a smaller incentive to continue. To see this, consider a threshold $\bar{p}$ at which player 2 quits. Without information, player 1 wins with certainty whenever the belief reaches $\bar{p}$, and he receives a payoff of 1 (see Proposition 1 and Figure 1). However, when the sender pipets at $\bar{p}$, player 1 does not win with certainty. Instead, the belief is reflected downwards with some probability and the game continues. Thus, pipetting at $\bar{p}$ hurts player 1 and player 1's value at $\bar{p}$ is strictly smaller (see Figure 2 and note that the dashed blue line is strictly below 1 at $\bar{p}$). By contrast, pipetting at $\underline{p}$ does not affect player 1's value. Intuitively, at belief $\underline{p}$, player 1 is indifferent between continuing and quitting. Thus, player 1's value on the interval $[0, \underline{p}]$ is zero. Pipetting at any $p \leq \underline{p}$ does not affect player 1's value, since the sender simply randomizes over beliefs at which player 1's value is constant (see again Figure 2).[23] Overall, player 1's value from continuing given the pipetting strategy $\{D^L(\underline{p}), D^R(\bar{p})\}$ is strictly lower than without information. The analog holds for player 2.

As the following Lemma shows, $\{D^L(\underline{p}), D^R(\bar{p})\}$ is actually the *minmax* strategy for each player. Thus, in the unique MPE, the sender minimizes each player's incentives to continue.

Lemma 1. *Suppose that $\tau_1 = \inf\{t \geq 0 : p_t < \underline{p}\}$ and $\tau_2 = \inf\{t \geq 0 : p_t > \bar{p}\}$. Then, for $i = 1, 2$,*

$$\{D^L(\underline{p}), D^R(\bar{p})\} \in \arg \inf_{D \in \mathcal{D}} \sup_{\tau_i \in \mathcal{T}} E\left[\int_0^{\tau_i \wedge \tau_{-i}} e^{-(r+\lambda)t} u_i(p_t)\, dt + e^{-(r+\lambda)\tau_{-i}} \mathbb{1}\{\tau_{-i} < \tau_i\}\right], \quad (18)$$

i.e., the pipetting strategy $\{D^L(\underline{p}), D^R(\bar{p})\}$ is the sender's minmax strategy for each player i, given the stopping strategy for player $-i$. Moreover,

$$\{0, D^R(\bar{p})\} \in \arg \inf_{D \in \mathcal{D}} \sup_{\tau_1 \in \mathcal{T}} E\left[\int_0^{\tau_1 \wedge \tau_2} e^{-(r+\lambda)t} u_1(p_t)\, dt + e^{-(r+\lambda)\tau_2} \mathbb{1}\{\tau_2 < \tau_1\}\right], \quad (19)$$

i.e., $D^L = 0$ and $D^R(\bar{p})$ is a minmax strategy for player 1, and

$$\{D^L(\underline{p}), 0\} \in \arg \inf_{D \in \mathcal{D}} \sup_{\tau_2 \in \mathcal{T}} E\left[\int_0^{\tau_1 \wedge \tau_2} e^{-(r+\lambda)t} u_2(p_t)\, dt + e^{-(r+\lambda)\tau_1} \mathbb{1}\{\tau_1 < \tau_2\}\right], \quad (20)$$

[23] More specifically, given the pipetting strategy $D^L(\underline{p})$, player 1's value function must satisfy the Robin boundary condition $V_1'(\underline{p}) = \frac{1}{\underline{p}} V_1(\underline{p})$. But since it is optimal for player 1 to stop at $\underline{p}$, the value matching and smooth pasting conditions $V_1(\underline{p}) = V_1'(\underline{p}) = 0$ imply that the Robin boundary condition trivially holds. Thus, given that the player 1 stops at $\underline{p}$, pipetting does not affect player 1's value.

i.e., $D^L(\underline{p})$ and $D^R = 0$ is a minmax strategy for player 2.[24]

The strategy $\{0, D^R(\bar{p})\}$ is also a minmax strategy for player 1 because pipetting at $\underline{p}$ does not affect player 1's value, as described above. Hence, player 1's value (and hence his stropping strategy) is the same under $\{D^L(\underline{p}), D^R(\bar{p})\}$ and $\{0, D^R(\bar{p})\}$, taking τ_2 as given. The analog holds for player 2, for whom $\{D^L(\underline{p}), 0)\}$ and $\{D^L(\underline{p}), D^R(\bar{p})\}$ are both minmax strategies.

Disclosure Trap. Lemma 1 implies that the unique MPE is suboptimal for the sender from an ex-ante perspective. In particular, the sender may fall into a "disclosure trap," which leaves her worse off than in the no-information benchmark. Intuitively, whenever exogenous information causes the belief to move out of the region $[\underline{p}, \bar{p}]$, it is optimal for the sender to reveal information to prevent one of the players from quitting. But anticipating these disclosures, each player has a lower value from continuing. This may cause the game to end earlier, compared to when the sender discloses no information.

Proposition 3. *There exists a unique pair of beliefs $(\underline{p}_0^S, \bar{p}_0^S)$ with $\underline{p}_0^S < \bar{p}_0^S$, such that*

$$W'_{ni}\left(\underline{p}_0^S\right) = \frac{W_{ni}\left(\underline{p}_0^S\right)}{\underline{p}_0^S} \text{ and } W'_{ni}\left(\bar{p}_0^S\right) = -\frac{W_{ni}\left(\bar{p}_0^S\right)}{1-\bar{p}_0^S}. \tag{21}$$

If $(\underline{p}, \bar{p}) \subset (\underline{p}_0^S, \bar{p}_0^S)$, then the sender is strictly worse off in the unique MPE for all $p_{0-} \in (\underline{p}, \bar{p})$, compared to the no-information benchmark.

This result is in sharp contrast to Orlov et al. (2020). In that paper, the sender only faces a single receiver and benefits from persuasion even though she cannot commit to dynamic disclosures. Here, the sender may be strictly worse off.

Intuitively, the thresholds $(\underline{p}_0^S, \bar{p}_0^S)$ arise from concavifying the sender's no information value. That is, for $p_{0-} > \bar{p}_0^S$, the sender randomizes between posteriors $p_0 = 1$ and $p_0 = \bar{p}_0^S$, and for $p_{0-} < \underline{p}_0^S$, the sender randomizes between posteriors $p_0 = 0$ and $p_0 = \underline{p}_0^S$. At the thresholds, the value from randomization must be tangent to $W_{ni}(p)$, which implies the conditions in Equation (21). These conditions take a similar form to the Robin boundary conditions in the MPE (see Equations (16) and (17)), which then allows me to establish a comparison theorem. Given the same boundary condition, if the interval is smaller, i.e. $(\underline{p}, \bar{p}) \subset (\underline{p}_0^S, \bar{p}_0^S)$, the sender is worse off.

[24]Note that for different disclosure policies, e.g. $\{D^L(\underline{p}), D^R(\bar{p})\}$ vs $\{0, D^R(\bar{p})\}$, players best responses and therefore the thresholds $\underline{p}$ and $\bar{p}$ are different in equilibrium. Lemma 1 is not concerned with characterizing equilibria, only with constructing minmax strategies for the sender. These minmax strategies are then used in Propositions 4 and 6, which construct Nash equilibria for players 1 and 2 given these strategies.

4.3 Commitment

I now contrast the unique MPE with the sender's commitment solution. With commitment, the sender chooses a (not necessarily Markovian) disclosure policy $D \in \mathcal{D}$. Taking D as given, each player i chooses the stopping strategy $\tau_i \in \mathcal{T}$ to be a best response to τ_{-i} and D. Equivalently, the commitment solution is given by

$$W(p_{0-}) = \sup_{D \in \mathcal{D}, \tau_1, \tau_2 \in \mathcal{T}} E\left[\int_0^{\tau_1 \wedge \tau_2} e^{-(r+\lambda)t} w dt\right] \tag{22}$$

subject to the incentive compatibility conditions

$$\tau_i \in \arg\sup_{\hat{\tau}_i \in \mathcal{T}} E\left[\int_0^{\hat{\tau}_i \wedge \tau_{-i}} e^{-(r+\lambda)t} u_i(p_t) dt + e^{-(r+\lambda)\tau_{-i}} \mathbb{1}\{\tau_{-i} < \hat{\tau}_i\}\right] \tag{23}$$

for player $i = 1, 2$.[25]

For tractability, I focus on symmetric threshold strategies for the sender. A disclosure strategy is of *threshold* type if there exist two thresholds $\underline{p}_C$ and $\bar{p}_C$, such that (1) the game ends at the first time the belief hits either $\underline{p}_C$ or $\bar{p}_C$ for $t > 0$ and such that (2) the sender discloses no information before that time.[26] A threshold strategy is *symmetric* if $\underline{p}_C + \bar{p}_C = 1$ and if

$$V_1(\underline{p}_C) = V_2(\bar{p}_C) \text{ and } V_1(\bar{p}_C) = V_2(\underline{p}_C). \tag{24}$$

A threshold strategy is *feasible* if

$$\min_{p \in [\underline{p}_C, \bar{p}_C]} V_i(p) \geq 0 \tag{25}$$

for $i = 1, 2$.

Intuitively, under a threshold strategy, the sender discloses nothing (except possibly for an initial disclosure at $t = 0$) until one of the thresholds $\underline{p}_C$ or $\bar{p}_C$ is reached. Then, either the sender provides no information and one of the players quits, or the sender provides a one-time disclosure, so that one player quits for any realized posterior. Thus, the game ends with certainty at $\underline{p}_C$ and $\bar{p}_C$.

Threshold strategies are appealing, since the sender promises valuable information in the future, to incentivize players to continue today. Such incentives feature prominently in the

[25] Note that the expectation in Equation (23) depends implicitly on the disclosure policy D.

[26] I allow the sender to disclose information at $t = 0$, i.e. $D_0^L > 0$ or $D_0^R > 0$ to concavity her value function. Since this is a one-time disclosure at $t = 0$, it does not affect player 1 and 2's incentives beyond changing the belief.

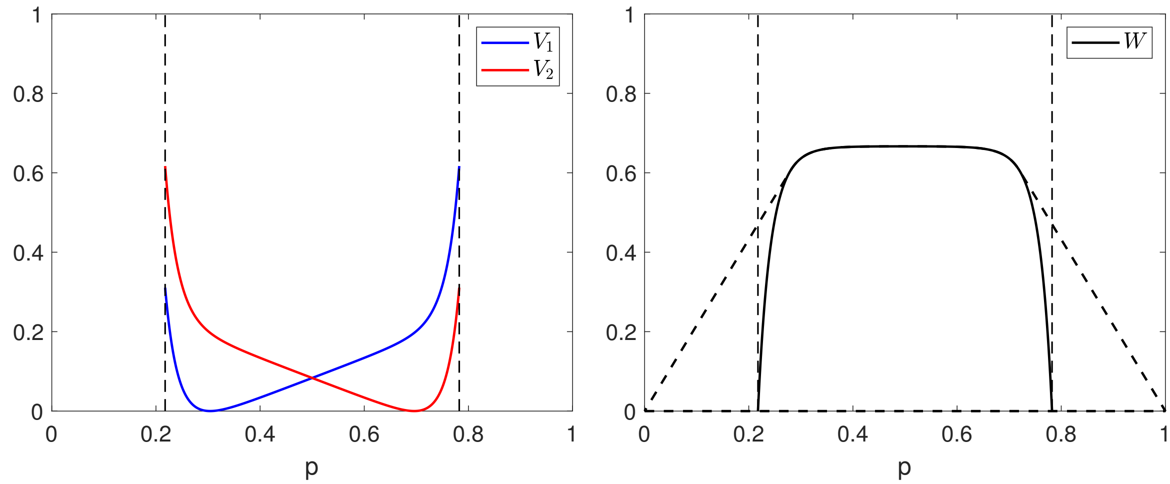

Figure 4: Optimal symmetric threshold strategy. The left panel illustrates the values of player 1 and 2 under the optimal threshold policy for $t > 0$, conditional on $p_0 \in (\underline{p}, \bar{p})$. In the right panel, the solid line is the sender's value under the threshold policy for $t > 0$ and the dashed lines illustrate the concavification of the sender's value, which is achieved at time $t = 0$ via the policies D_0^L and D_0^R in Equations (28) and (29). Parameters: $\lambda = 1.5$, $r = 1.5$, $c = 0.5$, $\sigma = 3$, $w = 2$.

literature on dynamic persuasion.[27]

Restricting attention to symmetric strategies is sensible given that player 1 and 2's flow payoffs are symmetric, i.e. $u_2(p) = u_1(1-p)$. Indeed, the unique MPE is symmetric, since in the MPE $V_2(p) = V_1(1-p)$ for any $p \in [0,1]$, and $\underline{p} + \bar{p} = 1$.[28] Condition (24) requires that at $\underline{p}_C$ and $\bar{p}_C$, the disclosure policies are symmetric, i.e. player 1's expected value given the disclosure at $\underline{p}_C$ equals player 2's expected value at $\bar{p}_C$. The analog holds for $\bar{p}_C$. Finally, Condition (25) requires that no player finds it optimal to stop for any belief $p \in [\underline{p}_C, \bar{p}_C]$.

Since the sender has commitment, she can threaten an arbitrary continuation strategy to induce players to quit. This commitment is valuable, since it gives the sender flexibility to induce different values for players 1 and 2 at the thresholds $\underline{p}_C$ and $\bar{p}_C$. For example, by providing disclosure that increases p and then threatening the minmax strategy for player 2 (Equation (20)), the sender can provide value to player 1. In other words, even though having a player quit at an interior belief $p \in (0,1)$ is ex-post suboptimal for the sender,[29] it is ex-ante beneficial.

In line with this intuition, the lemma below characterizes Nash equilibria between players 1 and 2, conditional on the minmax strategies in Lemma 1.

[27] See e.g. Ely and Szydlowski (2020), Smolin (2021), and Ely et al. (2021).

[28] See the proof of Proposition 2, especially Lemma 13.

[29] It is ex-post suboptimal, since the sender can always pipet information to make the game continue with positive probability.

Lemma 2. *There exists a pair of thresholds* $(\underline{p}_{1,mm}, \bar{p}_{1,mm})$*, so that given the strategy* $\{0, D^R(\bar{p}_{1,mm})\}$*, player 1 quits at* $\underline{p}_{1,mm}$ *and player 2 quits at* $\bar{p}_{1,mm}$*, and* $\underline{p}_{1,mm}$ *is the largest such threshold. There exists a pair of thresholds* $(\underline{p}_{2,mm}, \bar{p}_{2,mm})$*, so that given the strategy* $\{D^L(\underline{p}_{2,mm}), 0\}$*, player 1 quits at* $\underline{p}_{2,mm}$ *and player 2 quits at* $\bar{p}_{2,mm}$*, and* $\bar{p}_{2,mm}$ *is the smallest such threshold.*

Intuitively, $\underline{p}_{1,mm}$ is the largest belief at which player 1 quits, given any continuation strategy for the sender. Similarly, $\bar{p}_{2,mm}$ is the smallest belief at which player 2 quits, given any continuation strategy. The proposition below characterizes the optimal symmetric threshold strategy, which uses the thresholds defined in Lemma 2. Figure 4 illustrates the value functions.

Proposition 4. *The optimal symmetric threshold strategy is as follows. If* $p_{t-} = \underline{p}_C$*, the sender chooses* D_t^L *such that*

$$\frac{1 - D_t^L}{1 - D_{t-}^L} = \frac{\underline{p}_C}{1 - \underline{p}_C} \frac{1 - \bar{p}_{2,mm}}{\bar{p}_{2,mm}} \tag{26}$$

and $dD_t^R = 0$*, which induces posterior beliefs* $p_t \in \{0, \bar{p}_{2,mm}\}$*. At* $p_t = 0$*, player 1 quits and at* $p_t = \bar{p}_{2,mm}$*, the sender implements the continuation strategy* $\{D^L(\underline{p}_{2,mm}), 0\}$ *and player 2 quits. Similarly, at* $p_{t-} = \bar{p}_C$*, the sender chooses* D_t^R *such that*

$$\frac{1 - D_t^R}{1 - D_{t-}^R} = \frac{\underline{p}_{1,mm}}{1 - \underline{p}_{1,mm}} \frac{1 - \bar{p}_C}{\bar{p}_C} \tag{27}$$

and $dD_t^L = 0$*, which induces posteriors* $p_t \in \{\underline{p}_{1,mm}, 1\}$*. At* $p_t = \underline{p}_{1,mm}$*, the sender implements the continuation strategy* $\{0, D^R(\bar{p}_{1,mm})\}$ *and player 1 quits, and at* $p_t = 1$ *player 2 quits. The thresholds* $\underline{p}_C$ *and* $\bar{p}_C$ *satisfy*

$$\min_{p \in [\underline{p}_C, \bar{p}_C]} V_1(p) = \min_{p \in [\underline{p}_C, \bar{p}_C]} V_2(p) = 0.$$

Under this policy, the sender's value function $W(p)$ *satisfies the ODE (13) with boundary conditions* $W(\underline{p}_C) = W(\bar{p}_C) = 0$*. The sender's value is strictly concave for* $p \in (\underline{p}_C, \bar{p}_C)$*. There exist two thresholds* $(\underline{\pi}, \bar{\pi}) \subset (\underline{p}_C, \bar{p}_C)$*, so that whenever* $p_{0-} < \underline{\pi}$*,*

$$D_0^L = \frac{\underline{\pi} - p_{0-}}{\underline{\pi}(1 - p_{0-})} \text{ and } D_0^R = 0, \tag{28}$$

and whenever $p_{0-} > \bar{\pi}$

$$D_0^L = 0 \text{ and } D_0^R = \frac{p_{0-} - \bar{\pi}}{(1 - \bar{\pi})\, p_{0-}}. \tag{29}$$

Under the optimal policy, the sender delays disclosure until either $\underline{p}_C$ or $\bar{p}_C$ is reached. At $\underline{p}_C$, the sender chooses D_t^L such that conditional on not disclosing $v = 0$, the belief jumps to $\bar{p}_{2,mm}$. Then, the sender implements $\{D^L(\underline{p}_{2,mm}), 0\}$, the minmax policy for player 2, as the continuation policy, which induces player 2 to quit. Similarly, at $\bar{p}_C$, the sender chooses D_t^R such that conditional on not disclosing $v = 1$, the belief jumps to $\underline{p}_{1,mm}$. Then, the sender implements $\{0, D^R(\bar{p}_{1,mm})\}$, the minmax policy for player 1, and player 1 quits. The thresholds $\underline{p}_C$ and $\bar{p}_C$ are chosen such that the feasibility constraint (25) binds for each player. Otherwise, the sender can improve by widening the interval $(\underline{p}_C, \bar{p}_C)$ to $(\underline{p}_C - \varepsilon, \bar{p}_C + \varepsilon)$ for some $\varepsilon > 0$.

In choosing disclosure policies at the thresholds $\underline{p}_C$ and $\bar{p}_C$, the sender faces a tradeoff regarding player 1's and player 2's values. When the sender chooses D_t^L to induce posterior $\bar{p}_{2,mm}$, she could alternatively induce some posterior $\bar{p}' > \bar{p}_{2,mm}$. For any $\bar{p}' > \bar{p}_{2,mm}$, player 2 quits given the minmax policy. Inducing $\bar{p}'$ decreases $V_1(\underline{p}_C)$, but it increases $V_2(\underline{p}_C)$. However, as I show in the proof, such a policy is suboptimal. By lowering $\bar{p}'$, the sender can make player 1's feasibility constraint slack, i.e. $\min_{p \in [\underline{p}_C, \bar{p}_C]} V_1(p) > 0$ and improve her payoff. Intuitively, player 1 stopping decision depends more on his value at $\underline{p}_C$ than player 2's stopping decision depends on his value at $\underline{p}_C$. This is because the point at which player 1 is indifferent between continuing and stopping is closer to $\underline{p}_C$ (see Figure 4) and the point at which player 2 is indifferent is close to $\bar{p}_C$. Hence, increasing the value at $\underline{p}_C$ but decreasing the value at $\bar{p}_C$ increases player 1's incentives to continue. The analog holds for player 2.

This logic also illustrates why providing delayed full disclosure, i.e. choosing two thresholds $\underline{p}_{FD}$ and $\bar{p}_{FD}$ and then fully disclosing v once either threshold is reached, is suboptimal. Such a policy corresponds to choosing D_t^L so that $\bar{p}' = 1$ whenever $p_{t-} = \underline{p}_{FD}$, and the previous argument shows that this policy can be improved by lowering player 1's value at $\underline{p}_F$. Thus, unlike in Orlov et al. (2020), providing delayed full disclosure is not optimal.

Finally, whenever $p_{0-} < \underline{\pi}$ or $p_{0-} > \bar{\pi}$, it is optimal for the sender to provide instantaneous disclosure at time $t = 0$. This follows immediately from the shape of the sender's value $W(p)$, which is concave, hump-shaped, and strictly positive for $p \in (\underline{p}_C, \bar{p}_C)$, and zero otherwise.

4.4 Partisan Sender

In the main model, the sender wants to prolong the war of attrition and does not care about who wins. In this section, the sender instead favors player 1, and chooses disclosures to maximize the likelihood that player 1 wins. This leads to strikingly different predictions. In the unique MPE, the sender instantly reveals v, which is due her lack of commitment. With commitment, the sender provides instantaneous noisy disclosure, so that either player 1 or player 2 quits. Thus, in either case, the game ends instantaneously.

The sender's value for some given strategies $\tau_1, \tau_2 \in \mathcal{T}$ is now given by

$$W(p_{0-}) = \sup_{D \in \mathcal{D}} E\left[\int_0^{\tau_1 \wedge \tau_2} e^{-(r+\lambda)t} \lambda p_t dt + e^{-(r+\lambda)\tau_2} \mathbb{1}\{\tau_2 < \tau_1\}\right], \tag{30}$$

i.e. the sender receives a payoff of 1 whenever player 1 wins, either because player 2 quits or at time τ_E when $v = 1$. The following proposition characterizes the unique MPE.

Proposition 5. *There exists a unique MPE. In equilibrium, the sender truthfully reveals v at $t = 0$.*

To gain intuition, consider an MPE in which player 2 quits at some threshold $\bar{p} < 1$. If the sender does not disclose any information for $p < \bar{p}$, then her value is strictly increasing and strictly convex.[30] Because of this, her value for $p < \bar{p}$ is dominated by randomizing over posterior beliefs 0 and $\bar{p}$. That is, for any $p_{t-} < \bar{p}$, the sender is better off by choosing D_t^L such that

$$\frac{1 - D_t^L}{1 - D_{t-}^L} = \frac{p_{t-}}{1 - p_{t-}} \frac{1 - \bar{p}}{\bar{p}}.$$

This strategy induces posteriors $p_t \in \{0, \bar{p}\}$ and concavifies the sender's value for $p < \bar{p}$. But given this strategy, player 2 finds it optimal to continue at $\bar{p}$. If player 2 waits for some small amount of time, and the belief p_t drops below $\bar{p}$, then the sender reveals valuable information, and this information may lead player 2 to win instantly. This renders stopping suboptimal. Hence, in any such MPE, player 2 has a profitable deviation. The only MPE without such a deviation satisfies $\bar{p} = 1$. Then, since the posterior is degenerate, waiting for future information has no value for player 2. As I show in the proof, this is indeed the unique MPE. For any $p_{0-} \in (0, 1)$, the sender chooses $D_0^L = 1$ and hence instantly reveals v.

With commitment, the sender can improve. At $\bar{p}$, she can commit to the minmax strategy (Equation (20)) and induce player 2 to quit. As in Proposition 4, I consider threshold strategies. Since the sender only wants player 1 to win, I do not impose symmetry. Figure 5 illustrates the values given the optimal strategy.

Proposition 6. *The optimal threshold strategy is as follows. For $p_{0-} \geq \bar{p}_{2,mm}$,[31] the sender plays the minmax strategy $\{D^L(\underline{p}_{2,mm}), 0\}$ and player 2 quits immediately, while for $p_{0-} < \bar{p}_{2,mm}$, the sender chooses*

$$D_0^L = \frac{\bar{p}_{2,mm} - p_{0-}}{\bar{p}_{2,mm}(1 - p_{0-})}, \tag{31}$$

[30] This is an illustrative example. In the proof, I consider general strategies and show that they are all dominated by randomizing between 0 and $\bar{p}$.

[31] The thresholds $\underline{p}_{2,mm}$ and $\bar{p}_{2,mm}$ are characterized in Lemma 2.

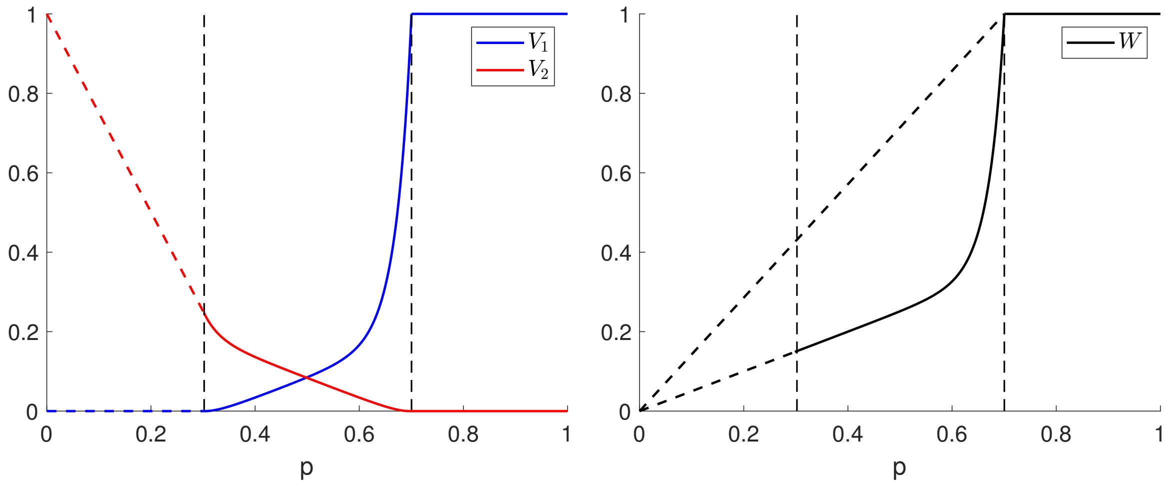

Figure 5: Optimal Threshold Strategy with Favoritism. The left panel illustrates the values of player 1 and 2 under the minmax policy $\{D^L(\underline{p}_{2,mm}), 0\}$. In the right panel, the solid line illustrates the sender's value without disclosure, and the dashed line connecting the points $(0,0)$ and $(\bar{p}_{2,mm}, 1)$ is the concavification of the sender's value, which is achieved by D_0^L in Equation (31). Parameters: $\lambda = 1.5$, $r = 1.5$, $c = 0.5$, $\sigma = 3$, $w = 2$.

and $D_0^R = 0$, *which induces posteriors* $p_0 \in \{0, \bar{p}_{2,mm}\}$. *If* $p_0 = 0$, *player 1 quits and if* $p_0 = \bar{p}_{2,mm}$, *the sender plays the minmax strategy* $\{D^L(\underline{p}_{2,mm}), 0\}$ *and player 2 quits.*

Unlike in Proposition 4, the sender does not prefer to wait until $\bar{p}_{2,mm}$ is reached. Instead, whenever $p_{0-} < \bar{p}_{2,mm}$, the sender instantaneously provides information to move the posterior to $\bar{p}_{2,mm}$. Thus, unlike in Proposition 4, the game ends instantaneously. The logic for this result is similar to the one in Proposition 5. If the sender discloses no information, her value is increasing and convex, and hence dominated by randomizing between posteriors 0 and $\bar{p}_{2,mm}$. The thresholds $\underline{p}_{2,mm}$ and $\bar{p}_{2,mm}$ are constructed so that given the minmax strategy $\{D^L(\underline{p}_{2,mm}), 0\}$, player 2 quits at $\bar{p}_{2,mm}$ and player 1 wins with certainty. Intuitively, this strategy minimizes $\bar{p}_{2,mm}$, by minimizing player 2's value from continuing and maximizing player 1's value from continuing. Thus, $\bar{p}_{2,mm}$ is the lowest threshold at which player 2 is willing to quit, given any continuation policy. This renders $\bar{p}_{2,mm}$ optimal for the sender.

5 Conclusion

This paper studies Bayesian persuasion in a war of attrition with exogenous information. The model sheds light on the use of rhetoric and information to incite conflict. In particular, the sender employs shifting rhetoric. Whenever exogenous information makes one player appear too strong, the sender provides information to make that player appear weaker. Conversely, whenever exogenous information makes one player appear too weak, the sender provides information to make that player appear stronger. On path, the sender alternates between making a player appear weak and strong, depending on the history of exogenous information.

I document a novel spillover effect, which is due to the sender facing two receivers. When the sender provides information to persuade one player to continue, she weakens the other player's incentives. This may lead the sender into a disclosure trap. Her efforts at prolonging the game weaken both players' incentives to continue, which leads the game to end earlier than if the sender were not able to disclose anything. With commitment, no such trap arises. Then, the sender provides delayed noisy information and threatens players with their minmax strategies to persuade them to quit. Such quitting is ex-post suboptimal for the sender, but allows her to prolong the game from an ex-ante perspective.

Finally, when the sender is partisan, and only cares about one player winning, the game ends instantaneously. In that case, without commitment, the sender provides full disclosure.

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A Proofs

A.1 Proof of Proposition 1

I apply the characterization of optimal stopping problems developed in Dayanik and Karatzas (2003). The proof proceeds as follows. (1) I characterize player 1's problem assuming that player 2 stops at some arbitrary threshold $\bar{p}$. (2) I show that player 2's problem is equivalent to player 1's problem, after suitably transforming the state p. (3) I show that there exists a unique fixed point, so that if player 2 stops at $\bar{p}$ and player 1 stops at $\underline{p}$, then stopping at $\bar{p}$ is indeed optimal for player 2 and stopping at $\underline{p}$ is indeed optimal for player 1. (4) Finally, I characterize the sender's value function in the unique no-information equilibrium.

Player 1's Problem. Fix $\bar{p} \geq 1 - c/\lambda$ and consider the following auxiliary problem of player 1,

$$V_1(p_{0-}) = \sup_{\tau_1 \in \mathcal{T}} E\left[\int_0^{\tau_1 \wedge \tau_2(\bar{p})} e^{-(r+\lambda)t} u_1(p_t)\,dt + e^{-(r+\lambda)\tau_2(\bar{p})} \mathbb{1}\{\tau_1 > \tau_2(\bar{p})\}\right], \tag{32}$$

where $\tau_2(\bar{p}) = \inf\{t \geq 0 : p_t > \bar{p}\}$. This problem is equivalent to

$$V_1(p_{0-}) = \frac{\lambda p_{0-} - c}{r+\lambda} + \tilde{V}_1(p_{0-}),$$

where[32]

$$\tilde{V}_1(p_{0-}) = \sup_{\tau_1 \in \mathcal{T}} E\left[e^{-(r+\lambda)\tau_1} g_1(p_{\tau_1})\,\mathbb{1}\{\tau_1 \leq \tau_2(\bar{p})\} + e^{-(r+\lambda)\tau_2(\bar{p})} l_1\left(p_{\tau_2(\bar{p})}\right)\mathbb{1}\{\tau_1 > \tau_2(\bar{p})\}\right],$$

and where

$$g_1(p) = -\frac{\lambda p - c}{r+\lambda} \text{ and } l_1(p) = 1 + g_1(p).$$

Let

$$h_1(p) = \begin{cases} g_1(p) & \text{if } p < \bar{p} \\ l_1(p) & \text{if } p \geq \bar{p} \end{cases}$$

and note that for $p \geq \bar{p}$, stopping immediately is optimal for player 1, since $l_1(p) > g_1(p)$ for $p \geq \bar{p}$.

Let $\mathcal{L}$ denote the infinitesimal generator of the diffusion process (2), i.e.,

$$\mathcal{L}u(p) = \frac{1}{2}\sigma^2(p)\,u''(p) \tag{33}$$

for any $u \in C^2([0,1])$. The ODE

$$(\mathcal{L} - (r+\lambda))\,u(p) = 0 \tag{34}$$

has one strictly increasing solution $\psi(p)$ and one strictly decreasing solution $\phi(p)$, which are unique

[32] The weak inequality $\tau_1 \leq \tau_2(\bar{p})$ in the first term is due to the tie-breaking rule, i.e. if $\tau_1 = \tau_2$ both players get zero.

up to multiplication (see e.g. Borodin and Salminen (2015), Chapter 2). They are given by

$$\psi(p) = p^{\frac{1}{2}(1+A)}(1-p)^{\frac{1}{2}(1-A)} \tag{35}$$

and

$$\phi(p) = p^{\frac{1}{2}(1-A)}(1-p)^{\frac{1}{2}(1+A)}, \tag{36}$$

where $A = \sqrt{1+8(r+\lambda)\sigma^2} > 1$. Note that $\phi(0) = \infty$ and $\psi(0) = 0$, and $\phi(1) = 0$ and $\psi(1) = \infty$, so both $p = 0$ and $p = 1$ are natural boundaries.

Let

$$F(p) \equiv \frac{\psi(p)}{\phi(p)} = \left(\frac{p}{1-p}\right)^A \tag{37}$$

and note that

$$F^{-1}(y) = \frac{1}{1+y^{-\frac{1}{A}}} \tag{38}$$

for $y \in [0,\infty)$. Define

$$H_1(y) = \frac{h_1\left(F^{-1}(y)\right)}{\phi\left(F^{-1}(y)\right)}, \tag{39}$$

and define $G_1(y)$ and $L_1(y)$ analogously, i.e.

$$G_1(y) = \frac{g_1\left(F^{-1}(y)\right)}{\phi\left(F^{-1}(y)\right)} \text{ and } L_1(y) = \frac{l_1\left(F^{-1}(y)\right)}{\phi\left(F^{-1}(y)\right)}.$$

Direct calculation yields

$$G_1(y) = -\frac{1}{r+\lambda}\left((\lambda-c)\,y^{\frac{1}{2}\left(1+\frac{1}{A}\right)} - cy^{\frac{1}{2}\left(1-\frac{1}{A}\right)}\right) \tag{40}$$

and

$$L_1(y) = G_1(y) + y^{\frac{1}{2}\left(1+\frac{1}{A}\right)}\left(1+y^{-\frac{1}{A}}\right). \tag{41}$$

In particular,

$$\begin{aligned}
G_1'(y) &= -\frac{1}{r+\lambda}\frac{1}{2A}y^{-\frac{1}{2}\left(1-\frac{1}{A}\right)}\left((\lambda-c)(A+1) - c(A-1)\,y^{-\frac{1}{A}}\right)\\
G_1''(y) &= -\frac{1}{r+\lambda}\frac{(A+1)(A-1)}{4A^2}y^{-\left(\frac{3}{2}-\frac{1}{2A}\right)}\left(-(\lambda-c)+cy^{-\frac{1}{A}}\right)\\
L_1'(y) &= \frac{r+c}{r+\lambda}\frac{1}{2}\left(1+\frac{1}{A}\right)y^{-\frac{1}{2}\left(1-\frac{1}{A}\right)} + \frac{r+\lambda+c}{r+\lambda}\frac{1}{2}\left(1-\frac{1}{A}\right)y^{-\frac{1}{2}\left(1+\frac{1}{A}\right)}\\
L_1''(y) &= -\frac{1}{r+\lambda}\frac{(A+1)(A-1)}{4A^2}\left((r+c)\,y^{-\left(\frac{3}{2}-\frac{1}{2A}\right)} + (r+\lambda+c)\,y^{-\left(\frac{3}{2}+\frac{1}{2A}\right)}\right).
\end{aligned}$$

The following properties of $G_1(y)$ and $L_1(y)$ follow from differentiating and using algebra.

Lemma 3. *The functions $G_1(y)$ and $L_1(y)$ in Equations (40) and (41) satisfy the following properties.*

- $G_1(0) = 0$, $G_1(\infty) = -\infty$

- $G_1(y) < 0$ *if and only if* $y > \left(\frac{c}{\lambda-c}\right)^A$
- $G_1'(y) < 0$ *if and only if* $y > \left(\frac{c}{\lambda-c}\frac{A-1}{A+1}\right)^A$
- $G_1''(y) > 0$ *if and only if* $y > \left(\frac{c}{\lambda-c}\right)^A$
- $G_1'''(y) > 0$ *if and only if* $y > \left(\frac{c}{\lambda-c}\frac{3+\frac{1}{A}}{3-\frac{1}{A}}\right)^A > \left(\frac{c}{\lambda-c}\right)^A$
- $G_1(y)$ *has a unique global maximum at* $y_{max} < \left(\frac{c}{\lambda-c}\right)^A$
- $L_1(0) = 0,\ L_1(\infty) = \infty$
- $L_1'(y) > 0,\ L_1''(y) < 0,\ L'''(y) > 0,$ *and* $L_1'(\infty) = 0,\ L_1''(\infty) = 0$
- $L_1(y) > G_1(y)$ *for* $y > 0$
- $L_1'(y) > G_1'(y)$
- $L_1''(y) < G_1''(y)$.

Since $\bar{p} \geq 1 - \frac{c}{\lambda} > \frac{c}{\lambda}$, we have $\bar{y} \geq F\left(\frac{c}{\lambda}\right) = \left(\frac{c}{\lambda-c}\right)^A > y_{max}$. Given the shapes of G_1 and L_1 characterized above, there exists a unique point $y_1^* \in (0, y_{max})$ such that the line through the points $(y_1^*, G_1(y_1^*))$ and $(\bar{y}, L_1(\bar{y}))$ is tangent to $G_1(y)$ at y_1^*, i.e.,

$$G_1'(y_1^*) = \frac{L_1(\bar{y}) - G_1(y_1^*)}{\bar{y} - y_1^*}. \tag{42}$$

Lemma 4. *For a fixed threshold* $\bar{p} \geq 1 - \frac{c}{\lambda}$*, it is optimal for player 1 to stop whenever* $p_t \leq \underline{p}$*, where* $\underline{p} = F^{-1}(y_1^*)$.

Proof. The function

$$\hat{H}_1(y) = \begin{cases} G_1(y) & \text{for } y < y_1^* \\ \frac{L_1(\bar{y}) - G_1(y_1^*)}{\bar{y} - y_1^*} y + \frac{\bar{y} G_1(y_1^*) - y_1^* L_1(\bar{y})}{\bar{y} - y_1^*} & \text{for } y_1^* \leq y \leq \bar{y} \\ L_1(y) & \text{for } y > \bar{y} \end{cases}$$

is the smallest nonnegative concave majorant of $H_1(y)$. This follows because $L_1(y) > G_1(y)$ for any $y > 0$, because $L_1(y)$ is strictly concave, and because $G_1(y)$ is strictly concave for $y \leq y_{max}$.[33] Dayanik and Karatzas (2003), Prop. 5.12 and Prop. 5.13, imply that the optimal stopping time is determined by characterizing the smallest nonnegative concave majorant of $H_1(y)$. Thus, it is optimal to stop for player 1 for all $p \leq \underline{p}$, where $\underline{p} \equiv F^{-1}(y_1^*)$. □

The Lemma below characterizes the value function $\tilde{V}_1(p)$.

[33] See Figure 6 for an illustration.

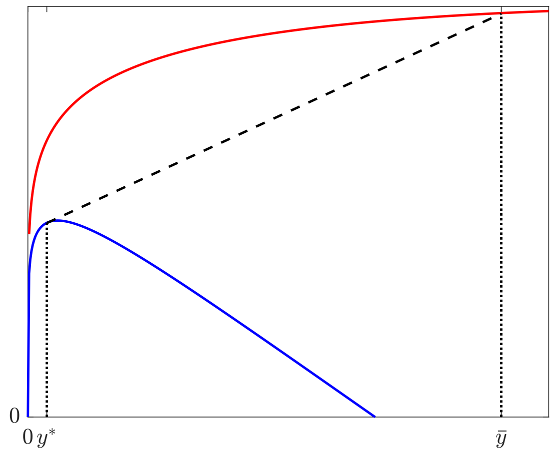

Figure 6: Illustration of $G_1(y)$ (blue), $L_1(y)$ (red), and $\hat{H}_1(y)$ (dashed).

Lemma 5. *For a fixed $\bar{p} \geq 1 - \frac{c}{\lambda}$, the value function $\tilde{V}_1(p)$ is twice continuously differentiable for $p \in (\underline{p}, \bar{p})$, with $\tilde{V}_1'(p) > 0$ and $\tilde{V}_1''(p) > 0$, and continuously differentiable at $p = \underline{p}$, so that*

$$\tilde{V}_1(\underline{p}) = g_1(\underline{p})$$

and the smooth pasting condition

$$\tilde{V}_1'(\underline{p}) = g_1'(\underline{p})$$

holds. At $\bar{p}$, $\tilde{V}_1(p)$ satisfies the value matching condition

$$\tilde{V}_1(\bar{p}) = l_1(\bar{p}).$$

Proof. Dayanik and Karatzas (2003), Prop. 5.12 implies that $\tilde{V}_1(p) = \phi(p)\hat{H}_1(F(p))$. In particular, for $p \in (\underline{p}, \bar{p})$, or equivalently $y \in (y_1^*, \bar{y})$, we have

$$\tilde{V}(p) = p^{\frac{1}{2}(1+A)}(1-p)^{\frac{1}{2}(1-A)} \frac{L_1(F(\bar{p})) - G_1(F(\underline{p}))}{F(\bar{p}) - F(\underline{p})},$$

which immediately implies that $\tilde{V}(p)$ is twice continuously differentiable and that $\tilde{V}'(p) > 0$ and $\tilde{V}''(p) > 0$.

Equation (42) implies that

$$G_1'(F(\underline{p})) = \frac{L_1(F(\bar{p})) - G_1(\underline{p})}{F(\bar{p}) - F(\underline{p})}.$$

For $y < y_1^*$ (or equivalently $p < \underline{p}$), we have $\hat{H}_1(y) = G_1(y)$ and

$$\tilde{V}_1(p) = \phi(p) G_1(F(p)) = g_1(p).$$

Thus,[34]

$$g_1'\left(\underline{p}\right)=\frac{d^-}{dp}\tilde{V}_1\left(p\right)\bigg|_{p=\underline{p}}=\frac{d^+}{dp}\tilde{V}_1\left(p\right)\bigg|_{p=\underline{p}}=\frac{L_1\left(F\left(\bar{p}\right)\right)-G_1\left(\underline{p}\right)}{F\left(\bar{p}\right)-F\left(\underline{p}\right)}$$

and in particular $\tilde{V}_1'\left(\underline{p}\right)=g_1'\left(\underline{p}\right)$. Finally, $\hat{H}_1\left(\bar{y}\right)=L_1\left(\bar{y}\right)$, which implies that $\tilde{V}_1\left(\bar{p}\right)=l_1\left(\bar{p}\right)$ after some algebra. □

The following Corollary follows immediately from the identity $V(p_{0-})=\frac{\lambda p_{0-}-c}{r+\lambda}+\tilde{V}_1(p_{0-})$.

Corollary 1. *$V(p_{0-})$ satisfies the ODE (11) with boundary conditions $V_1(\underline{p})=V_1'(\underline{p})=0$ and $V_1(\underline{p})=1$, and is strictly increasing and strictly convex.*

The following Lemma characterizes how y_1^* depends on $\bar{y}$, which is used later to establish that a unique Nash equilibrium.

Lemma 6. *We have*

$$\frac{dy_1^*}{d\bar{y}}>0 \text{ and } \frac{dy_1^*}{d\bar{y}}\frac{1}{y_1^{*2}}<1.$$

Proof. The implicit function theorem implies that

$$\frac{dy_1^*}{d\bar{y}}=-\frac{G_1'\left(y_1^*\right)-L_1'\left(\bar{y}\right)}{G_1''\left(y_1^*\right)\left(\bar{y}-y_1^*\right)}.$$

Since $y_1^*<y_{max}$, we have

$$G_1'\left(y_1^*\right)=\frac{L_1\left(\bar{y}\right)-G_1\left(y_1^*\right)}{\bar{y}-y_1^*}>L_1'\left(\bar{y}\right),$$

otherwise, the line starting from the point $\left(y_1^*,G_1\left(y_1^*\right)\right)$ with slope $G_1'\left(y_1^*\right)$ cannot possibly cross the function $L_1\left(y\right)$ at $y=\bar{y}$ from below. This implies that $dy_1^*/d\bar{y}>0$.

We have $dy_1^*/d\bar{y}<y_1^{*2}$ whenever

$$G_1'\left(y_1^*\right)-L_1'\left(\bar{y}\right)+y_1^{*2}G_1''\left(y_1^*\right)\left(\bar{y}-y_1^*\right)<0.$$

Since $L_1'\left(\bar{y}\right)>0$, a sufficient condition is that

$$G_1'\left(y_1^*\right)+y_1^{*2}G_1''\left(y_1^*\right)\left(\bar{y}-y_1^*\right)<0,$$

or equivalently since $y_1^*>0$ and $G_1''\left(y_1^*\right)<0$,

$$\frac{G_1'\left(y_1^*\right)}{G_1''\left(y_1^*\right)}+y_1^{*2}\left(\bar{y}-y_1^*\right)>0.$$

For any $y\in[0,y_{max}]$ and fixed $\bar{y}>y_{max}$, define the function

$$z\left(y\right)=\frac{G_1'\left(y\right)}{G_1''\left(y\right)}+y^2\left(\bar{y}-y\right).$$

[34] Here, $\frac{d^-}{dp}$ indicates the left derivative and $\frac{d^+}{dp}$ indicates the right derivative.

We have $z(y_{max}) > 0$, since $G_1'(y_{max}) = 0$ and we have $\lim_{y\to 0} z(y) = 0$, since $\lim_{y\to 0} G_1'(y)/G_1''(y) = 0$, which follows from algebra. Moreover,

$$z'(y) \quad = \quad 1 - y^2 - G_1'''(y)\, G_1'(y) / \left(G_1''(y)\right)^2 + 2y\,(\bar{y} - y)\,.$$

We have $G_1'''(y) < 0$ for all $y \leq y_{max}$, and $y \leq y_{max} < \left(\frac{c}{\lambda - c}\right)^A < 1$, which follows from our assumption that $\lambda > 2c$. Thus, $z(y) > 0$ for all $y \in (0, y_{max}]$, which establishes the result. $\square$

The following Lemma is used later in the construction of minmax strategies, in the proof of Lemma 1.

Lemma 7. *Consider the following variant of player 1's auxiliary problem in Equation (32),*

$$V_1(p_{0-}) = \sup_{\tau_1 \in \mathcal{T}} E\left[\int_0^{\tau_1 \wedge \tau_2(\bar{p})} e^{-(r+\lambda)t} u_1(p_t)\, dt + e^{-(r+\lambda)\tau_2(\bar{p})} \hat{V} \mathbb{1}\{\tau_1 > \tau_2(\bar{p})\}\right]$$

for some $\hat{V} > 0$. Then, $\underline{p}$ is strictly decreasing in $\hat{V}$.

Proof. Define $l_1(p) = \hat{V} + g_1(p)$ and $L_1(y) = G_1(y) + \hat{V} y^{\frac{1}{2}\left(1+\frac{1}{A}\right)}\left(1 + y^{-\frac{1}{A}}\right)$. The modified function $L_1(y)$ satisfies all properties in Lemma 3. Thus, there exists a unique y_1^* such that Equation (42) holds. Using the implicit function theorem with Equation (42) implies that $dy_1^*/d\hat{V} < 0$, or equivalently, $d\underline{p}/d\hat{V} < 0$. $\square$

Player 2's Problem. The proof proceeds analogously for player 2. Fix $\underline{p} \leq c/\lambda$ and consider the following auxiliary problem of player 2, i.e.,

$$V_2(p_{0-}) = \sup_{\tau_2 \in \mathcal{T}} E\left[\int_0^{\tau_2 \wedge \tau_1(\underline{p})} e^{-(r+\lambda)t} u_2(p_t)\, dt + e^{-(r+\lambda)\tau_1(\underline{p})} \mathbb{1}\{\tau_2 > \tau_1(\underline{p})\}\right],$$

where $\tau_1(\underline{p}) = \inf\{t \geq 0 : p_t < \underline{p}\}$. This problem is equivalent to

$$\begin{aligned} V_2(p_{0-}) \quad &= \quad \frac{\lambda(1 - p_{0-}) - c}{r + \lambda} + \sup_{\tau_2 \in \mathcal{T}} E\Big[e^{-(r+\lambda)\tau_2} g_2(p_{\tau_2})\, \mathbb{1}\left\{\tau_2 \leq \tau_1(\underline{p})\right\} \\ &\qquad + e^{-(r+\lambda)\tau_1(\underline{p})} l_2\left(p_{\tau_1(\underline{p})}\right) \mathbb{1}\left\{\tau_2 > \tau_1(\underline{p})\right\}\Big], \end{aligned}$$

where

$$g_2(p) = -\frac{\lambda(1-p) - c}{r + \lambda} \text{ and } l_2(p) = 1 + g_2(p)\,.$$

Define

$$h_2(p) = \begin{cases} g_2(p) & \text{if } p > \underline{p} \\ l_2(p) & \text{if } p \leq \underline{p}. \end{cases}$$

Player 2's problem can be transformed as follows. Let $\hat{p}_t = 1 - p_t$ and note that $d\hat{p}_t = -\sigma(\hat{p}_t)\, d\hat{B}_t$ by Ito's Lemma. Equivalently,

$$d\hat{p}_t = \sigma(\hat{p}_t)\, d\tilde{B}_t \tag{43}$$

where $\tilde{B}_t = -\hat{B}_t$ is a Brownian motion. Thus, player 2's problem is equivalent to

$$\begin{aligned} V_2(\hat{p}_0) &= \frac{\lambda \hat{p}_0 - c}{r+\lambda} + \sup_{\tau_2 \in \mathcal{T}} \hat{E}\left[e^{-(r+\lambda)\tau_2} g_2(1-\hat{p}_{\tau_2}) \mathbb{1}\left\{\tau_2 \le \tau_1\left(\hat{\bar{p}}\right)\right\} \right. \\ &\quad \left. + e^{-(r+\lambda)\tau_1(\hat{\bar{p}})} l_2\left(1 - p_{\tau_1(\hat{\bar{p}})}\right) \mathbb{1}\left\{\tau_2 > \tau_1\left(\hat{\bar{p}}\right)\right\}\right], \end{aligned}$$

where $\hat{\bar{p}} = 1 - \underline{p} > 1 - \frac{c}{\lambda}$ and where $\hat{E}[.]$ denotes the expectation given the changed state variable $\hat{p}_t$. We have

$$g_2(1-\hat{p}) = g_1(\hat{p}) \text{ and } l_2(1-\hat{p}) = l_1(\hat{p}),$$

so that

$$V_2(1-p) = V_1(p).$$

Define

$$h_2(\hat{p}) = \begin{cases} g_1(\hat{p}) & \text{if } p < \hat{\bar{p}} \\ l_1(\hat{p}) & \text{if } p \ge \hat{\bar{p}}. \end{cases}$$

The stochastic differential equation (43) admits the generator given in Equation (33), and thus for $u(p) \in C^2([0,1])$ the ODE

$$(\mathcal{L} - (r+\lambda))\, u(\hat{p}) = 0$$

admits an increasing solution $\psi(\hat{p})$ and a decreasing solution $\phi(\hat{p})$, which are given by Equations (35) and (36), respectively. Define $\hat{y} \equiv F(\hat{p})$, and note that $\hat{y} \in [0,\infty)$ and that

$$\hat{y} = \left(\frac{\hat{p}}{1-\hat{p}}\right)^A = \left(\frac{1-p}{p}\right)^A = \frac{1}{y}.$$

Define $\hat{\bar{y}} = F\left(\hat{\bar{p}}\right)$ and define

$$\hat{G}_2(\hat{y}) = \frac{g_2\left(1 - F^{-1}(\hat{y})\right)}{\phi\left(F^{-1}(\hat{y})\right)}$$

and

$$\hat{L}_2(\hat{y}) = \frac{l_2\left(1 - F^{-1}(\hat{y})\right)}{\phi\left(F^{-1}(\hat{y})\right)},$$

so that

$$\hat{H}_2(\hat{y}) = \begin{cases} \hat{G}_2(\hat{y}) & \text{if } p < \hat{\bar{y}} \\ \hat{L}_2(\hat{y}) & \text{if } p \ge \hat{\bar{y}}. \end{cases}$$

As for player 1, Dayanik and Karatzas (2003), Prop. 5.12 and Prop. 5.13, imply that the optimal stopping time is determined by characterizing the smallest nonnegative concave majorant of $\hat{H}_2(\hat{y})$. But note that

$$\hat{G}_2(\hat{y}) = \frac{g_1\left(F^{-1}(\hat{y})\right)}{\phi(\hat{y})} = G_1(\hat{y})$$

and

$$\hat{L}_2(\hat{y}) = \frac{l_1\left(F^{-1}(\hat{y})\right)}{\phi\left(F^{-1}(\hat{y})\right)} = L_1(\hat{y}).$$

Hence, given $\hat{\hat{y}}$, player 2's problem is identical to player 1's problem. In particular, there exists a unique point $\hat{y}_2^*$, so that

$$G_1'\left(\hat{y}_2^*\right) = \frac{L_1\left(\hat{\hat{y}}\right) - G_1\left(\hat{y}_2^*\right)}{\hat{\hat{y}} - \hat{y}_2^*}$$

and it is optimal for player to stop whenever $\hat{p} \leq \underline{\hat{p}}$, where $\underline{\hat{p}} \equiv F^{-1}\left(\hat{y}_2^*\right)$. Equivalently, it is optimal for player 2 to stop whenever $p \geq \bar{p}$, where $\bar{p} \equiv 1 - \underline{\hat{p}}$, so that $\tau_2^* = \inf\left\{t \geq 0 : p_t \geq \bar{p}\right\}$.

Since player 1's problem and player 2's problem are identical, it holds that

$$\frac{d\hat{y}_2^*}{d\hat{\hat{y}}} \in \left(0, \hat{y}_2^*\right).$$

In particular, the proof of Lemma 6 goes through without modifications. Finally, $V_2(p)$ satisfies the following properties.

Lemma 8. *For a fixed $\underline{p} \leq \frac{c}{\lambda}$, the value function $V_2(p)$ is twice continuously differentiable for $p \in (\underline{p}, \bar{p})$, with $V_2'(p) < 0$ and $V_2''(p) > 0$, and continuously differentiable at $p = \bar{p}$, so that*

$$V_2(\bar{p}) = 0$$

and the smooth pasting condition

$$V_2'(\bar{p}) = 0$$

holds. At $\underline{p}$, $V_2(p)$ satisfies the value matching condition

$$V_2\left(\underline{p}\right) = 1.$$

The proof is analogous to the proof of Lemma 5 and Corollary 1, using the fact that

$$V_2(1-p) = V_1(p).$$

Any Nash Equilibrium is in Threshold Strategies. Consider an arbitrary Nash equilibrium in pure strategies, so that player 1 stops if $p_t \in S_1$ and player 2 stops if $p_t \in S_2$. Since the boundaries $p_t = 0$ and $p_t = 1$ are natural, and since $\lambda \cdot 0 - c < 0$ and $\lambda(1-1) - c < 0$, we have $0 \in S_1$ and $1 \in S_2$. For player 1, stopping is suboptimal whenever $\lambda p_t \geq c$ and thus $S_1 \subset [0, \frac{c}{\lambda})$. For player 2, stopping is suboptimal whenever $\lambda(1-p_t) \geq c$ and thus $S_2 \subset (1 - \frac{c}{\lambda}, 1]$. Now, let $\bar{p} \equiv \inf S_2 \geq 1 - \frac{c}{\lambda}$ and note that whenever $p_{0-} < \bar{p}$, we have $p_t \leq \bar{p}$ for any $t < \tau_1 \wedge \tau_2 \wedge \tau_E$. Then, Lemma 4 implies that player 1's optimal stopping region is of the form $S_1 = [0, \underline{p}]$ for some $\underline{p} < \frac{c}{\lambda}$. For any $p > 1 - \frac{c}{\lambda}$ we have $\lambda p > c$. Thus, stopping at such a p is suboptimal for player 1 and hence $p \notin S_1$, and in particular, $S_1 \cap S_2 = \emptyset$. Thus, for any S_2, player 1's optimal stopping region is of the form $S_1 = [0, \underline{p}]$. An analog argument establishes that for any $S_1 \subset [0, \frac{c}{\lambda})$, player 2's optimal stopping region is of the form $S_2 = [\bar{p}, 1]$. Thus, any pure Nash equilibrium is in threshold strategies.

Existence and Uniqueness. This section establishes that there exists a unique pair $(\underline{p}, \bar{p})$, so that given $\bar{p}$, it is optimal for player 1 to stop at $\underline{p}$, and given $\underline{p}$, it is optimal for player 2 to stop at

$\bar{p}$. Fix a $\bar{p} \geq 1 - \frac{c}{\lambda}$ and define

$$\bar{y} = F(\bar{p}) \in Y \equiv \left[\left(\frac{\lambda - c}{c} \right)^A, \infty \right).$$

Let y_1^* denote player 1's optimal stopping point in y-space given $\bar{y}$. Define $\hat{\bar{y}} = 1/y_1^*$, and given $\hat{\bar{y}}$ define $\hat{y}_2^*$ as player 2's optimal stopping point in $\hat{y}$-space. Finally, given $\hat{y}_2^*$, define $\bar{y}' = 1/\hat{y}_2^*$ (in y-space). Note that $\hat{y}_2^* < \left(\frac{c}{\lambda - c}\right)^A$ and hence $\bar{y}' > \left(\frac{\lambda-c}{c}\right)^A$ so that $\bar{y}' \in Y$ whenever $\bar{y} \in Y$. Denote the resulting function with $\bar{Y}(\bar{y})$ and note that $\bar{Y} : Y \to Y$ and that $\bar{Y}(\bar{y})$ is continuously differentiable.

Given this chain of transformations, we have

$$\begin{aligned} \frac{d\bar{Y}}{d\bar{y}} &= \frac{dy_1^*}{d\bar{y}} \cdot \frac{d\hat{\bar{y}}}{y_1^*} \cdot \frac{d\hat{y}_2^*}{d\hat{\bar{y}}} \cdot \frac{d\bar{y}'}{d\hat{y}_2^*} \\ &= \frac{dy_1^*}{d\bar{y}} \cdot \frac{1}{y_1^{*2}} \cdot \frac{d\hat{y}_2^*}{d\hat{\bar{y}}} \cdot \frac{1}{\hat{y}_2^{*2}} \\ &< 1, \end{aligned}$$

which follows from the fact that

$$\frac{dy_1^*}{d\bar{y}} \frac{1}{y_1^{*2}} < 1 \text{ and } \frac{d\hat{y}_2^*}{d\hat{\bar{y}}} \frac{1}{\hat{y}_2^{*2}} < 1.$$

Thus, the function $\bar{Y}(\bar{y})$ has slope strictly below one and can cross the identity line at most once. If $\bar{y} = \left(\frac{\lambda-c}{c}\right)^A$, then $y_2^* < \left(\frac{c}{\lambda-c}\right)^A$ and $\bar{Y} > \left(\frac{\lambda-c}{c}\right)^A$, and hence

$$\bar{Y}\left(\left(\frac{\lambda - c}{c}\right)^A\right) > \left(\frac{\lambda - c}{c}\right)^A.$$

Letting $\bar{y} \to \infty$ implies that $y_1^* \to y_{max} < \left(\frac{c}{\lambda-c}\right)^A$, which in turn implies that y_2^* is bounded away from zero, so that $\bar{Y} = 1/y_2^* < \infty$. Thus, $\lim_{\bar{y}\to\infty} \bar{Y}(\bar{y})/\bar{y} = 0 < 1$. Hence, there exists a unique fixed point $\bar{y}'(\bar{y}^{**}) = \bar{y}^{**}$. We can define $\bar{p}_{ni} = F^{-1}(\bar{y}^{**})$ and $\underline{p}_{ni} = F^{-1}(y_1^*)$, where y_1^* is the solution to player 1's problem given $\bar{y}^{**}$.[35]

Sender's Value Function. The following Lemma characterizes the sender's value function given the unique fixed point $(\underline{p}_{ni}, \bar{p}_{ni})$.

Lemma 9. *For $(\underline{p}_{ni}, \bar{p}_{ni}) \subset (0,1)$, the ODE (13) with boundary conditions $W_{ni}(\underline{p}_{ni}) = W_{ni}(\bar{p}_{ni}) = 0$ admits a unique solution. This solution is strictly concave and equals the sender's value function in Equation (10) for $D^L = D^R = 0$.*

Proof. The proof uses the method of lower and upper solutions (e.g. De Coster and Habets (2006), Th. 1.5, p. 81). A twice continuously differentiable function $\alpha(p)$ is a lower solution of Equation

[35]Here, note that $F(p)$ is a bijection.

(13) if

$$\alpha''(p) \geq \frac{2}{\sigma(p)^2}\left((r+\lambda)\alpha(p) - w\right) \tag{44}$$

for $p \in (\underline{p}, \bar{p})$, $\alpha(\underline{p}) \leq 0$, and $\alpha(\bar{p}) \leq 0$. A twice continuously differentiable function $\beta(p)$ is an upper solution of Equation (13) if

$$\beta''(p) \leq \frac{2}{\sigma(p)^2}\left((r+\lambda)\beta(p) - w\right)$$

for $p \in (\underline{p}, \bar{p})$, $\beta(\underline{p}) \geq 0$, and $\beta(\bar{p}) \geq 0$.

A lower solution of Equation (13) is given by $\alpha(p) = -Me^{p}$, for some M such that $M > 0$ and

$$M \leq \inf_{p \in [\underline{p}, \bar{p}]} \frac{2}{\sigma(p)^2} w e^{-p} \frac{1}{\max\left\{1 - \frac{2(r+\lambda)}{\sigma(p)^2}, 0\right\}}.$$

Here, $1/0 = \infty$ to save notation. Specifically, if $1 - \frac{2(r+\lambda)}{\sigma(p)^2} < 0$, then Equation (44) is trivially satisfied at p for any $M > 0$. An upper solution of Equation (13) is given by $\beta(p) = \frac{w}{r+\lambda}$ for $p \in [\underline{p}, \bar{p}]$.

De Coster and Habets (2006), Th. 1.5, p. 81, states that ODE (13) admits a twice continuously differentiable solution with boundary conditions $W_{ni}(\underline{p}_{ni}) = W_{ni}(\bar{p}_{ni}) = 0$ if it admits a lower and an upper solution and if the function

$$f(p, u) = \frac{2}{\sigma(p)^2}\left((r+\lambda)u - w\right)$$

is continuous for $p \in [\underline{p}, \bar{p}]$ and $u \in [\alpha(p), \beta(p)]$. Both conditions are satisfied.

The following argument establishes uniqueness. Take two solutions $W(p)$ and $\tilde{W}(p)$ satisfying the ODE (13) and the boundary conditions, i.e. $W(\underline{p}) = \tilde{W}(\underline{p}) = 0$ and $W(\bar{p}) = \tilde{W}(\bar{p}) = 0$. Let $Z(p) = W(p) - \tilde{W}(p)$. We have $Z(\underline{p}) = Z(\bar{p}) = 0$ and $Z(p)$ satisfies the ODE

$$(r+\lambda) Z(p) = \frac{1}{2}\sigma(p)^2 Z''(p). \tag{45}$$

Suppose that $Z(p)$ has a strictly positive maximum for some $p \in (\underline{p}, \bar{p})$. Then, we must have $Z(p) > 0$ and $Z''(p) = 0$. Equation (45) implies that this is a contradiction. Similarly, suppose that $Z(p)$ has a strictly negative minimum for some $p \in (\underline{p}, \bar{p})$. Then, we must have $Z(p) < 0$ and $Z''(p) = 0$, another contradiction. Thus, the unique solution to Equation (45) which satisfies $Z(\underline{p}) = Z(\bar{p}) = 0$ is $Z(p) = 0$ for all $p \in [\underline{p}, \bar{p}]$. This implies that $W(p) = \tilde{W}(p)$, i.e. the solution to Equation (45) with boundary conditions $W_{ni}(\underline{p}) = W_{ni}(\bar{p}) = 0$ is unique.

It remains to provide a verification argument, which establishes that the solution to Equation (45) equals the sender's value function. Let $W_{ni}(p)$ denote the solution to Equation (13). Since this function is twice continuously differentiable on $[\underline{p}, \bar{p}]$, applying Ito's formula to $e^{-(r+\lambda)t} W_{ni}(p_t)$

for $t < \tau_1 \wedge \tau_2$ yields

$$\begin{aligned} W_{ni}(p_{0-}) &= E\left[\int_0^{\tau_1\wedge\tau_2} e^{-(r+\lambda)t}\left(\frac{1}{2}W_{ni}''(p_t)\,\sigma(p_t)^2 - (r+\lambda)W_{ni}(p_t)\right)dt\right] \\ &= E\left[\int_0^{\tau_1\wedge\tau_2} we^{-(r+\lambda)t}dt\right]. \end{aligned}$$

Thus, the solution to Equation (13) equals the sender's no-information value in Equation (10).

Finally, the following argument establishes concavity. Denote with $\bar{W}_{ni}(p)$ the sender's value when both players never stop. This value is given by $\bar{W}_{ni}(p) = \frac{w}{r+\lambda}$, and clearly $\bar{W}_{ni}(p) > W_{ni}(p)$ for all $p \in (0,1)$. Then, Equation (13) implies that

$$\frac{1}{2}\sigma(p)^2 W_{ni}''(p) = (r+\lambda)W_{ni}(p) - w < (r+\lambda)\bar{W}_{ni}(p) - w = 0$$

for any $p \in [\underline{p}, \bar{p}]$. □

A.2 Proof of Proposition 2

The proof proceeds as follows. (1) I characterize player 1's stopping problem given a right-pipetting strategy $D^R(\bar{p})$ (but no left-pipetting, i:e. $D^L = 0$) and player 2's problem given a left-pipetting strategy $D^L(\underline{p})$ (but no right-pipetting, i.e. $D^R = 0$). (2) I show that these solutions satisfy a fixed-point property, i.e. there exists a unique pair $(\underline{p}, \bar{p})$, so that given strategy $D^R(\bar{p})$ player 1 stops at $\underline{p}$, and given strategy $D^L(\underline{p})$ player 2 stops at $\bar{p}$. (3) I show that given these stopping strategies, the pipetting strategy $\{D^L(\underline{p}), D^R(\bar{p})\}$ is the sender's best response. (4) I verify that player 1 and 2's values constructed in steps (1) and (2) remain unchanged given the strategy $\{D^L(\underline{p}), D^R(\bar{p})\}$. This establishes that the tuple $\{D^L(\underline{p}), D^R(\bar{p}), \tau_1(\underline{p}), \tau_2(\bar{p})\}$ is an MPE. (5) I show that no other MPE can exist.

Verification. The following verification argument for pipetting strategies is used throughout. Let $u(p)$ be continuous on $[0,1]$. Given a fixed pipetting strategy $\{D^L(\underline{p}), D^R(\bar{p})\}$ with $0 < \underline{p} < \bar{p} < 1$ and a pair of fixed payoffs (V_ξ, V_η), define

$$V(p_{0-}) = \sup_{\tau\in\mathcal{T}} E\left[\int_0^{\tau\wedge\tau_\eta\wedge\tau_\xi} e^{-(r+\lambda)t}u(p_t)\,dt + e^{-(r+\lambda)\tau_\xi}V_\xi \mathbb{1}\{\tau_\xi \le \tau_1\wedge\tau_\eta\} + e^{-(r+\lambda)\tau_\eta}V_\eta \mathbb{1}\{\tau_\eta \le \tau_1\wedge\tau_\xi\}\right],$$

where τ_ξ and τ_η are defined in Equation (8).

Proposition 7. *Suppose that there exists a function $f(p)$ with $f \in C^2((0,1)) \setminus \{\underline{p}, \bar{p}\}$ and $f \in C^1((0,1))$. Define $S = \{p \in [\underline{p}, \bar{p}] : f(p) = 0\}$ and suppose that*

(i) $f(p) \ge 0$ for all $p \in [0,1]$,

(ii) $(r+\lambda)f(p) = \mathcal{L}f(p) + u(p)$ for $p \in [\underline{p}, \bar{p}] \setminus S$,

(iii) $(r+\lambda)f(p) \ge \mathcal{L}f(p) + u(p)$ for $p \in S$,

(iv) $f'(\bar{p}) = \frac{V_\xi - f(\bar{p})}{1-\bar{p}}$ and $f'(\underline{p}) = \frac{f(\underline{p}) - V_\eta}{\underline{p}}$,

(v) $f(p) = \frac{1-p}{1-\bar{p}}f(\bar{p}) + \frac{p-\bar{p}}{1-\bar{p}}V_\xi$ for $p \in [\bar{p}, 1]$ and $f(p) = \frac{p}{\underline{p}}f(\bar{p}) + \frac{\underline{p}-p}{\underline{p}}V_\eta$ for $p \in [0, \underline{p}]$.

Then, $f(p_{0-}) = V(p_{0-})$.

Proof. For any almost surely finite stopping time τ, we have

$$
\begin{aligned}
& E\left[\int_0^{\tau\wedge\tau_\eta\wedge\tau_\xi} e^{-(r+\lambda)t} u\left(p_t\right) dt\right] \qquad (46)\\
&= E\left[\int_0^{\infty} e^{-(r+\lambda)t} u\left(p_t\right) \mathbb{1}\left\{\tau > t\right\} \mathbb{1}\left\{\tau_\xi > t\right\} \mathbb{1}\left\{\tau_\eta > t\right\} dt\right] \\
&= p_{0-} E\left[\int_0^{\infty} e^{-(r+\lambda)t} u\left(p_t\right) \mathbb{1}\left\{\tau > t\right\} \Pr\left(\tau_\xi > t | \mathcal{F}_t, v=1\right) \Pr\left(\tau_\eta > t | \mathcal{F}_t, v=1\right) dt\right] \\
&+ \left(1-p_{0-}\right) E\left[\int_0^{\infty} e^{-(r+\lambda)t} u\left(p_t\right) \mathbb{1}\left\{\tau > t\right\} \Pr\left(\tau_\xi > t | \mathcal{F}_t, v=0\right) \Pr\left(\tau_\eta > t | \mathcal{F}_t, v=0\right) dt\right] \\
&= p_{0-} E\left[\int_0^{\infty} e^{-(r+\lambda)t} u\left(p_t\right) \mathbb{1}\left\{\tau > t\right\} (1-D_t^R) dt\right] \\
&+ \left(1-p_{0-}\right) E\left[\int_0^{\infty} e^{-(r+\lambda)t} u\left(p_t\right) \mathbb{1}\left\{\tau > t\right\} (1-D_t^L) dt\right] \\
&= E\left[\int_0^{\tau} e^{-(r+\lambda)t} u\left(p_t\right) \pi\left(D_t\right) dt\right],
\end{aligned}
$$

where

$$
\pi\left(D_t\right) = p_{0-}\left(1-D_t^R\right) + \left(1-p_{0-}\right)\left(1-D_t^L\right). \qquad (47)
$$

Here, note that $\pi\left(D_{0-}\right) = 1$, that

$$
\Pr\left(\tau_\eta > t | \mathcal{F}_t, v=1\right) = 1 \text{ and } \Pr\left(\tau_\eta > t | \mathcal{F}_t, v=0\right) = 1 - D_t^L,
$$

and that

$$
\Pr\left(\tau_\xi > t | \mathcal{F}_t, v=1\right) = 1 - D_t^R \text{ and } \Pr\left(\tau_\xi > t | \mathcal{F}_t, v=0\right) = 1.
$$

In particular, we have $\tau_\xi = \infty$ if $v = 0$ and $\tau_\eta = \infty$ if $v = 1$, which implies that

$$
\begin{aligned}
E\left[e^{-(r+\lambda)\tau_\xi} \mathbb{1}\left\{\tau_\xi \leq \tau \wedge \tau_\eta\right\}\right] &= p_{0-} E\left[\int_0^{\infty} e^{-(r+\lambda)t} \mathbb{1}\left\{t < \tau\right\} d\Pr\left(\tau_\xi \leq t | \mathcal{F}_t, v=1\right)\right] \\
&= p_{0-} E\left[\int_0^{\tau} e^{-(r+\lambda)t} dD_t^R\right].
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
E\left[e^{-(r+\lambda)\tau_\eta} \mathbb{1}\left\{\tau_\eta \leq \tau \wedge \tau_\xi\right\}\right] &= \left(1-p_{0-}\right) E\left[\int_0^{\infty} e^{-(r+\lambda)t} \mathbb{1}\left\{t < \tau\right\} d\Pr\left(\tau_\eta \leq t | \mathcal{F}_t, v=0\right)\right] \\
&= \left(1-p_{0-}\right) E\left[\int_0^{\tau} e^{-(r+\lambda)t} dD_t^L\right].
\end{aligned}
$$

Thus,

$$\begin{aligned} V\left(p_{0-}\right) &= \sup_{\tau} E\Bigg[\int_{0}^{\tau} e^{-(r+\lambda)t} u\left(p_t\right)\pi\left(D_t\right)dt + p_{0-}V_{\xi}\int_{0}^{\tau} e^{-(r+\lambda)t} dD_t^R \\ &\quad + \left(1-p_{0-}\right)V_{\eta}\int_{0}^{\tau} e^{-(r+\lambda)t} dD_t^L\Bigg]. \end{aligned}$$

Define

$$\tilde{f}_t \equiv e^{-(r+\lambda)t}\pi\left(D_t\right)f\left(p_t\right).$$

Now, applying Harrison (2013), Prop. 4.14, p. 70, and Ito's Lemma for semi-martingales (Protter (2005), Th. II.32, p. 78) to the process $\tilde{f}_t$ yields

$$\begin{aligned} e^{-(r+\lambda)t}\pi_t f\left(p_t\right) - f\left(p_{0-}\right) &= \int_0^t e^{-(r+\lambda)s}\pi\left(D_s\right)\left(\left(\mathcal{L}f\left(p_s\right)-(r+\lambda)f\left(p_s\right)\right)ds + f'\left(p_s\right)\sigma\left(p_s\right)d\hat{B}_s\right) \\ &\quad + \int_0^t e^{-(r+\lambda)s}\left(-p_{0-}f\left(p_s\right) - f'\left(p_s\right)\pi\left(D_s\right)\frac{p_s\left(1-p_s\right)}{1-D_s^R}\right)dD_s^{R,c} \\ &\quad + \int_0^t e^{-(r+\lambda)s}\left(-\left(1-p_{0-}\right)f\left(p_s\right) + f'\left(p_s\right)\pi\left(D_s\right)\frac{p_s\left(1-p_s\right)}{1-D_s^L}\right)dD_s^{L,c} \\ &\quad + \sum_{0\le s\le t}\left(\pi\left(D_s\right)f\left(p_s\right) - \pi\left(D_{s-}\right)f\left(p_{s-}\right)\right). \end{aligned}$$

The last line sums over the jump times of the process p_t, i.e., $p_s \neq p_{s-}$ for all s in the sum. Equivalently, $D_s^R - D_{s-}^R + D_s^L - D_{s-}^L > 0$. In the equation above, $D_t^{R,c}$ is the continuous part of D_t^R and $D_t^{L,c}$ is the continuous part of D_t^L. By construction (see Equations (3)–(6)), D_t^R and D_t^L are continuous processes for all $t > 0$, i.e. $dD_t^R = dD_t^{R,c}$ and $dD_t^L = dD_t^{L,c}$, so that for all $t > 0$, $p_{t-} = p_t$, $D_{t-}^R = D_t^R$, and $D_{t-}^R = D_t^R$. Thus, it follows that

$$\sum_{0\le s\le t}\left(\pi\left(D_s\right)f\left(p_s\right) - \pi\left(D_{s-}\right)f\left(p_{s-}\right)\right) = \pi\left(D_0\right)f\left(p_0\right) - f\left(p_{0-}\right),$$

where $D_{0-}^R = D_{0-}^L = 0$, so that $\pi\left(D_{0-}\right) = 1$. Note that whenever $p_{0-} \in (\underline{p}, \bar{p})$, then $D_0^R = D_{0-}^R = 0$ and $D_0^L = D_{0-}^L = 0$ so that $p_0 = p_{0-}$ and $\pi\left(D_0\right) = 1$.

The standard localization argument (e.g. Øksendal (2003), Th. 10.4.1, p. 227ff) implies that for any almost surely finite stopping time τ, we have[36]

$$\begin{aligned} f\left(p_{0-}\right) &= E\left[\int_0^{\tau} e^{-(r+\lambda)t}\pi\left(D_t\right)\left((r+\lambda)f\left(p_t\right) - \mathcal{L}f\left(p_t\right)\right)dt\right] \\ &\quad + E\left[\int_0^{\tau} e^{-(r+\lambda)t}\left(p_{0-}f\left(p_t\right) + f'\left(p_t\right)\pi\left(D_t\right)\frac{p_t\left(1-p_t\right)}{1-D_t^R}\right)dD_t^{R,c}\right] \\ &\quad + E\left[\int_0^{\tau} e^{-(r+\lambda)t}\left(\left(1-p_{0-}\right)f\left(p_t\right) - f'\left(p_t\right)\pi\left(D_t\right)\frac{p_t\left(1-p_t\right)}{1-D_t^L}\right)dD_t^{L,c}\right] \\ &\quad + E\left[f\left(p_{0-}\right) - \pi\left(D_0\right)f\left(p_0\right)\right]. \end{aligned}$$

[36] Here, note that $f(p)$ is bounded, since it is continuous on a bounded domain.

Since p_t is a martingale, we have

$$
\begin{aligned}
p_{0-} &= p_{0-}D_t^R + p_t\left(p_{0-}\left(1-D_t^R\right) + \left(1-p_{0-}\right)\left(1-D_t^L\right)\right) \\
&= p_{0-}D_t^R + p_t\pi\left(D_t\right),
\end{aligned} \tag{48}
$$

which implies that

$$\pi\left(D_t\right) = \frac{p_{0-}\left(1-D_t^R\right)}{p_t}.$$

Since for $t \geq 0$, $p_t = \bar{p}$ whenever $dD_t^R > 0$, we have

$$p_{0-}f\left(p_t\right) + f'\left(p_t\right)\pi\left(D_t\right)\frac{p_t\left(1-p_t\right)}{1-D_t^R} = p_{0-}\left(f\left(\bar{p}\right) + f'\left(\bar{p}\right)\left(1-\bar{p}\right)\right),$$

and in particular

$$p_{0-}\left(f\left(\bar{p}\right) + f'\left(\bar{p}\right)\left(1-\bar{p}\right)\right) = p_{0-}V_\xi$$

by (iv). This implies that if $p_{0-} \in (\underline{p}, \bar{p})$, we have

$$E\left[\int_0^\tau e^{-(r+\lambda)t}\left(p_{0-}f\left(p_t\right) + f'\left(p_t\right)\pi\left(D_t\right)\frac{p_t\left(1-p_t\right)}{1-D_t^R}\right)dD_t^R\right] = p_{0-}V_\xi E\left[\int_0^\tau e^{-(r+\lambda)t}dD_t^R\right].$$

Similarly, we have $p_t = \underline{p}$ whenever $dD_t^L > 0$, and Equation (48) implies that

$$1 - p_t = \frac{\left(1-p_{0-}\right)\left(1-D_t^L\right)}{\pi\left(D_t\right)},$$

so that

$$\left(1-p_{0-}\right)f\left(p_t\right) - f'\left(p_t\right)\pi\left(D_t\right)\frac{p_t\left(1-p_t\right)}{1-D_t^L} = \left(1-p_{0-}\right)\left(f\left(\underline{p}\right) - f'\left(\underline{p}\right)\underline{p}\right),$$

and

$$\left(1-p_{0-}\right)\left(f\left(\underline{p}\right) - f'\left(\underline{p}\right)\underline{p}\right) = (1-p_{0-})V_\eta$$

by (iv). Thus, if $p_{0-} \in (\underline{p}, \bar{p})$, we have

$$E\left[\int_0^\tau e^{-(r+\lambda)t}\left(\left(1-p_{0-}\right)f\left(p_t\right) - f'\left(p_t\right)\pi\left(D_t\right)\frac{p_t\left(1-p_t\right)}{1-D_t^L}\right)dD_t^L\right] = \left(1-p_{0-}\right)E\left[\int_0^\tau e^{-(r+\lambda)t}dD_t^L\right].$$

For $p_{0-} \geq \bar{p}$, we have $dD_0^L = 0$ and $dD_0^R = D_0^R$ where

$$D_0^R = \frac{p_{0-} - \bar{p}}{1-\bar{p}}\frac{1}{p_{0-}},$$

which follows from Equation (3). Then,

$$f\left(p_{0-}\right)-\pi_{0} f\left(p_{0}\right)=f\left(p_{0-}\right)-\frac{1-p_{0-}}{1-\bar{p}} f\left(\bar{p}\right)=\frac{p_{0-}-\bar{p}}{1-\bar{p}} V_{\xi},$$

using Equation (3). But also,

$$\Pr\left(\tau_{\xi}=0 \mid \mathcal{F}_{0}, v=1\right)=D_{0}^{R}>0$$

and

$$E\left[e^{-(r+\lambda) \tau_{\xi}} \mathbb{1}\left\{\tau_{\xi} \leq \tau \wedge \tau_{\eta}\right\}\right] = p_{0-} E\left[\int_{0}^{\tau} e^{-(r+\lambda) t} d D_{t}^{R}\right]=p_{0-}\left(D_{0}^{R}+E\left[\int_{0}^{\tau} e^{-(r+\lambda) t} d D_{t}^{R, c}\right]\right).$$

Thus,

$$f\left(p_{0-}\right)-\pi_{0} f\left(p_{0}\right)=f\left(p_{0-}\right)-\frac{1-p_{0-}}{1-\bar{p}} f\left(\bar{p}\right)=\frac{p_{0-}-\bar{p}}{1-\bar{p}} V_{\xi}=V_{\xi} p_{0-} D_{0}^{R}.$$

For $p_{0-} \leq \underline{p}$, we have similarly $dD_{0}^{R}=0$ and $dD_{0}^{L}=D_{0}^{L}$, where

$$D_{0}^{L}=\frac{\underline{p}-p_{0-}}{\underline{p}(1-p_{0-})},$$

using Equation (5). Then, we have

$$f\left(p_{0-}\right)-\pi_{0} f\left(p_{0}\right)=f\left(p_{0-}\right)-\frac{p_{0-}}{\underline{p}} f(\underline{p})=\frac{\underline{p}-p_{0-}}{\underline{p}} V_{\eta}$$

and

$$\Pr\left(\tau_{\eta}=0 \mid \mathcal{F}_{0}, v=0\right)=D_{0}^{L},$$

and thus

$$\begin{aligned} E\left[e^{-(r+\lambda) \tau_{\eta}} \mathbb{1}\left\{\tau_{\eta} \leq \tau \wedge \tau_{\xi}\right\}\right] &= \left(1-p_{0-}\right)\left(D_{0}^{L}+E\left[\int_{0}^{\tau} e^{-(r+\lambda) t} d D_{t}^{L, c}\right]\right) \\ &= \frac{\underline{p}-p_{0-}}{\underline{p}}+\left(1-p_{0-}\right) E\left[\int_{0}^{\tau} e^{-(r+\lambda) t} d D_{t}^{L, c}\right]. \end{aligned}$$

Collecting equations and using the inequality $(r+\lambda) f(p) \geq \mathcal{L} f(p)+u(p)$ implies that

$$\begin{aligned} f\left(p_{0-}\right) \geq\ & E\left[\int_{0}^{\tau} e^{-(r+\lambda) t} u\left(p_{t}\right) \pi\left(D_{t}\right) d t\right]+p_{0-} V_{\xi} E\left[\int_{0}^{\tau} e^{-(r+\lambda) t} d D_{t}^{R}\right] \\ & +\left(1-p_{0-}\right) V_{\eta} E\left[\int_{0}^{\tau} e^{-(r+\lambda) t} d D_{t}^{L}\right]. \end{aligned}$$

Since τ was arbitrary, this implies that $f\left(p_{0-}\right) \geq V\left(p_{0-}\right)$.

Let $\tau_S = \inf\{t \geq 0 : t \in S\}$. Then, since $(r+\lambda) f(p) = \mathcal{L} f(p) + u(p)$ for $p \in [\underline{p}, \bar{p}] \setminus S$, we have

$$\begin{aligned} f(p_{0-}) &= E\left[\int_0^{\tau_S} e^{-(r+\lambda)t} u(p_t)\,\pi(D_t)\,dt\right] + p_{0-} V_\xi E\left[\int_0^{\tau_S} e^{-(r+\lambda)t} dD_t^R\right] \\ &\quad + (1-p_{0-}) V_\eta E\left[\int_0^{\tau_S} e^{-(r+\lambda)t} dD_t^L\right] \\ &\leq \sup_\tau E\left[\int_0^{\tau} e^{-(r+\lambda)t} u(p_t)\,\pi(D_t)\,dt\right] + p_{0-} V_\xi E\left[\int_0^{\tau} e^{-(r+\lambda)t} dD_t^R\right] \\ &\quad + (1-p_{0-}) V_\eta E\left[\int_0^{\tau} e^{-(r+\lambda)t} dD_t^L\right] \\ &= V(p_{0-}). \end{aligned}$$

Combining the two inequalities yields $f(p_{0-}) = V(p_{0-})$ for any $p_{0-} \in [0,1]$. □

Player 1's Auxiliary Problem. Consider player 1's problem for a given threshold $\bar{p} \geq 1 - \frac{c}{\lambda}$, so that the sender uses the right-pipetting strategy $D^R(\bar{p})$ and $D^L = 0$. Pick an arbitrary $\hat{V} > 0$ and define for $p \geq \bar{p}$ the pipetting value for player 1 as

$$V_1^{pip}(p, \hat{V}) = \frac{p - \bar{p}}{1 - \bar{p}} + \frac{1-p}{1-\bar{p}} \hat{V}.$$

This is the value player 1 gets when the sender randomizes between beliefs $\bar{p}$ and 1 and if the continuation value at $\bar{p}$ is some arbitrary number $\hat{V}$.

Consider the following auxiliary problem of player 1, for a given threshold $\bar{p}$ and for $p_{0-} < \bar{p}$,

$$V_1(p_{0-}) = \sup_{\tau_1} E\left[\int_0^{\tau_1 \wedge \tau_2(\bar{p})} e^{-(r+\lambda)t} u_1(p_t)\,dt + e^{-(r+\lambda)\tau_2(\bar{p})} V_1^{pip}\left(\bar{p}, \hat{V}\right) \mathbb{1}\{\tau_1 > \tau_2(\bar{p})\}\right]. \tag{49}$$

In this problem, player 1 chooses τ_1 and the game exogenously ends at the stopping time $\tau_2(\bar{p}) = \inf\{t \geq 0 : t \geq \bar{p}\}$. Thus, once p_t reaches $\bar{p}$, player 1 receives the continuation value $V_1^{pip}(\bar{p}, \hat{V}) = \hat{V}$ and the game ends. Clearly, this problem is not necessarily equivalent to player 1's original problem, in Equation (9). However, the two problems are equivalent for $p_{0-} < \bar{p}$ if the Robin boundary condition

$$\frac{d}{dp} V_1(p)\Big|_{p=\bar{p}} = \frac{d}{dp} V_1^{pip}(p, \hat{V})\Big|_{p=\bar{p}}$$

holds, which is proven below.

Player 1's auxiliary problem is equivalent to

$$V_1(p_{0-}) = \frac{\lambda p_{0-} - c}{r+\lambda} + \sup_{\tau_1} E\left[e^{-(r+\lambda)\tau_1} g_1(p_{\tau_1}) \mathbb{1}\{\tau_1 \leq \tau_2(\bar{p})\} + e^{-(r+\lambda)\tau_2(\bar{p})} l_1\left(p_{\tau_2(\bar{p})}\right) \mathbb{1}\{\tau_1 > \tau_2(\bar{p})\}\right], \tag{50}$$

where

$$g_1(p) = -\frac{\lambda p - c}{r+\lambda} \text{ and } l_1(p) = V_1^{pip}(p, \hat{V}) + g_1(p).$$

Define

$$\tilde{V}_1(p_{0-}) = V_1(p_{0-}) - \frac{\lambda p_{0-} - c}{r+\lambda} \tag{51}$$

as in the proof of Proposition 1. Given this transformation, the Robin boundary condition at $\bar{p}$ becomes

$$\begin{aligned} \frac{d}{dp}\tilde{V}_1(p)\Big|_{p=\bar{p}} &= \frac{d}{dp}V_1^{pip}(p,\hat{V})\Big|_{p=\bar{p}} + g_1'(\bar{p}) \\ &= \frac{1-\hat{V}}{1-\bar{p}} - \frac{\lambda}{r+\lambda}. \end{aligned} \tag{52}$$

Define

$$h_1(p) = \begin{cases} g_1(p) & \text{if } p < \bar{p} \\ l_1(p) & \text{if } p \geq \bar{p}, \end{cases}$$

define $y = F(p)$, where $F(p) = \left(\frac{p}{1-p}\right)^A$ just as in Equation (37), and define $G_1(y) = g_1\left(F^{-1}(y)\right)/\phi\left(F^{-1}(y)\right)$, $L_1(y) = l_1\left(F^{-1}(y)\right)/\phi\left(F^{-1}(y)\right)$, and

$$H_1(y) = \begin{cases} G_1(y) & \text{if } y < \bar{y} \\ L_1(y) & \text{if } y \geq \bar{y}, \end{cases}$$

where $\bar{y} = F(\bar{p})$ and where $\phi(p) = p^{\frac{1}{2}(1-A)}(1-p)^{\frac{1}{2}(1+A)}$, as in Equation (36). The function $G_1(y)$ is given by Equation (40) and satisfies the same properties as in Lemma 3, whereas

$$L_1(y) = \frac{r+c}{r+\lambda} y^{\frac{1}{2}\left(1+\frac{1}{A}\right)} + y^{\frac{1}{2}\left(1-\frac{1}{A}\right)}\left(K + \frac{c}{r+\lambda}\right), \tag{53}$$

where

$$K = \hat{V} - \bar{y}^{\frac{1}{A}}\left(1-\hat{V}\right).$$

We have

$$\begin{aligned} L_1'(y) &= \frac{A+1}{2A} y^{-\frac{1}{2}\left(1-\frac{1}{A}\right)} \frac{r+c}{r+\lambda} + \frac{A-1}{2A} y^{-\frac{1}{2}\left(1+\frac{1}{A}\right)}\left(K + \frac{c}{r+\lambda}\right) \\ L_1''(y) &= -\frac{(A+1)(A-1)}{4A^2} y^{-\left(\frac{3}{2}-\frac{1}{2A}\right)}\left(\frac{r+c}{r+\lambda} + y^{-\frac{1}{A}}\left(K + \frac{c}{r+\lambda}\right)\right). \end{aligned}$$

Note that K is differentiable in $\hat{V}$ and $\bar{y}$, and increasing in $\hat{V}$ and decreasing in $\bar{y}$.

Lemma 10. *The function $L_1(y)$ in Equation (53) satisfies the following properties.*

- $L_1(0) = 0$, $L_1(\infty) = \infty$, $L_1'(\infty) = 0$, $L_1''(\infty) = 0$
- *If $K \geq -\frac{c}{r+\lambda}$, then $L_1(y) > 0$, $L_1'(y) > 0$, $L_1''(y) < 0$ for all $y > 0$, and $L_1(y) > G_1(y)$ for all $y > (\max\{-K, 0\})^A$*
- *If $K < -\frac{c}{r+\lambda}$, then*

– $L_1(y) > 0$ *and* $L_1''(y) < 0$ *if and only if* $y > \left(-\frac{r+\lambda}{r+c}\left(K + \frac{c}{r+\lambda}\right)\right)^A$

– $L_1'(y) > 0$ *if and only if* $y > \left(-\frac{A-1}{A+1}\frac{r+\lambda}{r+c}\left(K + \frac{c}{r+\lambda}\right)\right)^A$

- *If* $K > 0$*, then* $L_1(y) > G_1(y)$*,* $L_1'(y) > G_1'(y)$*, and* $L_1''(y) < G_1''(y)$ *for all* $y > 0$
- *If* $K < 0$*, then*

  – $L_1(y) > G_1(y)$ *and* $L_1''(y) < G_1''(y)$ *if and only if* $y > (-K)^A$

  – $L_1'(y) > G_1'(y)$ *if and only if* $y > \left(-K\frac{A-1}{A+1}\right)^A$

- $\frac{d}{d\hat{V}}L_1(y) > 0$ *for all* $y > 0$ *and* $\frac{d}{d\bar{y}}L_1(y) < 0$ *for all* $y > 0$*.*

Since $L_1(y)$ has a qualitatively different shape depending whether K is positive, consider the two cases separately. Suppose first that $K \geq 0$. Then, $L_1(y)$ and $G_1(y)$ have qualitatively the same shapes as in the no-information benchmark, which are characterized in Lemma 3. Then, just as in the no-information benchmark (see Lemma 4), the smallest concave majorant of the function $H_1(y)$ is given by

$$\hat{H}_1(y) = \begin{cases} G_1(y) & \text{for } y < y_1^* \\ \frac{L_1(\bar{y})-G_1(y_1^*)}{\bar{y}-y_1^*}y + \frac{\bar{y}G_1(y_1^*)-y_1^*L_1(\bar{y})}{\bar{y}-y_1^*} & \text{for } y_1^* \leq y \leq \bar{y} \\ L_1(y) & \text{for } y > \bar{y} \end{cases}$$

for some $y_1^* < y_{max}$, such that

$$G_1'(y_1^*) = \frac{L_1(\bar{y}) - G_1(y_1^*)}{\bar{y} - y_1^*}.$$

Dayanik and Karatzas (2003), Prop. 5.12 and Prop. 5.13, imply that given the pair $(\bar{p}, \hat{V})$, it is optimal for player 1 to stop at $\underline{p} = F^{-1}(y_1^*)$. Moreover, player 1's optimal value function for the transformed problem in Equation (50) is given by

$$\tilde{V}_1^*(p) = \phi(p)\hat{H}_1(F(p)). \tag{54}$$

From this expression, it is immediate that $\tilde{V}_1^*(p)$ is differentiable in p whenever $\hat{H}_1(F(p))$ is differentiable in p, since both $\phi(p)$ and $F(p)$ are differentiable. We have $G_1'(y_1^*) > L_1'(\bar{y})$ since $L_1(y) > G_1(y)$ and since $L_1(y)$ is strictly concave (see Figure 6 for illustration). Therefore,

$$\frac{d^-}{dp}\hat{H}_1(F(p))\Big|_{p=\bar{p}} > \frac{d^+}{dp}\hat{H}_1(F(p))\Big|_{p=\bar{p}}.$$

For $p > \bar{p}$,

$$\phi(p)\hat{H}_1(F(p)) = \phi(p)L_1(F(p)) = l_1(p) = V_1^{pip}(p, \hat{V}) - \frac{\lambda p - c}{r + \lambda}.$$

Thus,

$$\frac{d^-}{dp}\tilde{V}_1^*(p)\Big|_{p=\bar{p}} > \frac{d}{dp}\left(V_1^{pip}(p, \hat{V}) - \frac{\lambda p - c}{r + \lambda}\right)\Big|_{p=\bar{p}},$$

which implies that the smooth pasting condition (52) cannot hold at $\bar{p}$ whenever $K \geq 0$.

Consider now the case $K < 0$. The smallest concave majorant of $H_1(y)$ is given by

$$\hat{H}_1(y) = \begin{cases} G_1(y) & \text{for } y < y_1^* \\ \frac{L_1(\tilde{y})-G_1(y_1^*)}{\tilde{y}-y_1^*}y + \frac{\tilde{y}G_1(y_1^*)-y_1^*L_1(\tilde{y})}{\tilde{y}-y_1^*} & \text{for } y_1^* \leq y \leq \tilde{y} \\ L_1(y) & \text{for } y > \tilde{y} \end{cases}$$

so that

$$G_1'(y_1^*) = \frac{L_1(\tilde{y}) - G_1(y_1^*)}{\tilde{y} - y_1^*} = L_1'(\tilde{y}). \tag{55}$$

Here $y_1^* < y_{max}$ and $y_1^* < \tilde{y}$, and given $G_1(y)$ and $L_1(y)$, the pair $(y_1^*, \tilde{y})$ is unique. In particular, since $K < 0$, we have $L_1(y_1^*) < G_1(y_1^*)$ and $L_1(\tilde{y}) > G_1(\tilde{y})$. The line connecting the points $(y_1^*, G_1(y_1^*))$ and $(\tilde{y}, L_1(\tilde{y}))$ is tangent to $G_1(y)$ at $y = y_1^*$ and is tangent to $L_1(y)$ at $y = \tilde{y}$. Moreover, the points y_1^* and $\tilde{y}$ depend continuously on $\hat{V}$.

Player 1's optimal value is given by the analog of Equation (54), i.e. $\tilde{V}_1^*(p) = \phi(p)\hat{H}_1(F(p))$. We generally have $\tilde{y} \neq \bar{y}$. If $\tilde{y} > \bar{y}$, the value matching condition

$$\tilde{V}_1^*(\bar{p}) = V_1^{pip}\left(\bar{p}, \hat{V}\right) - \frac{\lambda\bar{p} - c}{r + \lambda}$$

cannot hold, because we have $\hat{H}_1(\bar{y}) > L_1(\bar{y})$, which implies that

$$\tilde{V}_1^*(\bar{p}) = \phi(\bar{p})\hat{H}_1(F(\bar{p})) > \phi(\bar{p}) L_1(F(\bar{p})) = V_1^{pip}\left(\bar{p}, \hat{V}\right) - \frac{\lambda\bar{p} - c}{r + \lambda}.$$

The Lemma below establishes that there exists a unique $\hat{V} > 0$ so that $\tilde{y} = \bar{y}$.

Lemma 11. *There exists a unique*

$$\hat{V}^* \in \left(0, \frac{1}{1 + \bar{y}^{\frac{1}{A}}}\right)$$

such that $\tilde{y} = \bar{y}$. Given $\hat{V}^*$, the value matching condition

$$\tilde{V}_1^*(\bar{p}) = V_1^{pip}(\bar{p}, \hat{V}^*) - \frac{\lambda\bar{p} - c}{r + \lambda}$$

and the smooth pasting condition

$$\tilde{V}_1^{*\prime}(\bar{p}) = \frac{d}{dp} V_1^{pip}(p, \hat{V}^*) - \frac{\lambda p - c}{r + \lambda}\bigg|_{p=\bar{p}}$$

both hold.

Proof. Letting $\hat{V} \uparrow \frac{1}{1+\bar{y}^{\frac{1}{A}}}$ implies that $K \uparrow 0$. Recall that $L_1(y) > G_1(y)$ if and only if $y > (-K)^A$, so that the point at which $L_1(y)$ crosses $G_1(y)$ from below approaches 0. Since by construction $L_1((-K)^A) = G_1((-K)^A) > 0$, $L_1(y)$ is strictly concave for all $y > (-K)^A$. We also have $L_1'((-K)^A) > G_1'((-K)^A)$ and $G_1(y)$ is strictly concave on $(0, (-K)^A)$. This implies that there exists a unique pair $(y_1^*, \tilde{y})$ so that Equation (55) holds, where $y_1^* < (-K)^A < \tilde{y}$. As $K \uparrow 0$, $\tilde{y}$ is arbitrarily close to y_1^*, and in particular $\tilde{y} < \bar{y}$. Conversely, letting $\hat{V} \downarrow 0$ implies that $K \downarrow -\bar{y}^{\frac{1}{A}}$, so that $L_1(y)$ crosses $G_1(y)$ for the first time from below at $y = \bar{y}$. This implies that $\tilde{y} > \bar{y}$. Hence, since $\tilde{y}$ is continuous in $\hat{V}$, there exists a $\hat{V}^* \in (0, 1/(1+\bar{y}^{\frac{1}{A}}))$ such that $\tilde{y} = \bar{y}$ whenever $\hat{V} = \hat{V}^*$. Since $L_1'(y)$ is strictly increasing in $\hat{V}$, $\tilde{y}$ is strictly decreasing in $\hat{V}$. Thus, $\hat{V}^*$ is unique.

For $\hat{V} = \hat{V}^*$, the tangency condition in Equation (55) implies that

$$\tilde{V}_1^*(\bar{p}) = \phi(\bar{p})\hat{H}_1(F(\bar{p})) = \phi(\bar{p})L_1(F(\bar{p})) = V_1^{pip}(\bar{p}, \hat{V}^*) - \frac{\lambda\bar{p} - c}{r+\lambda}$$

and that

$$\begin{aligned}\tilde{V}_1^{*\prime}(\bar{p}) &= \frac{d^-}{dp}\phi(p)\hat{H}_1(F(p))\Big|_{p=\bar{p}} = \frac{d^+}{dp}\phi(p)\hat{H}_1(F(p))\Big|_{p=\bar{p}} \\ &= \frac{d}{dp}\phi(p)L_1(F(p))\Big|_{p=\bar{p}} = \frac{d}{dp}V_1^{pip}(p, \hat{V}^*) - \frac{\lambda p - c}{r+\lambda}\Big|_{p=\bar{p}}.\end{aligned}$$

□

The following Corollary to Lemma 11 will be used later. It is recorded here to not interrupt the flow of the argument.

Corollary 2. *As $\bar{y} \to 0$, $y_1^* \to 0$.*

Proof. Lemma 11 implies that for any $\bar{y} > 0$, there exists a $\hat{V} > 0$ and a $y_1^* < \bar{y}$, such that the double tangency condition (55) holds at $\tilde{y} = \bar{y}$. This immediately implies that $y_1^* \to 0$ as $\bar{y} \to 0$. □

The following Lemma provides a verification argument, based on Proposition 7.

Lemma 12. *Define $\tilde{V}_1^*(p_{0-})$ as*

$$\tilde{V}_1^*(p_{0-}) = \begin{cases} g_1(p_{0-}) & \text{if} \quad p_{0-} \le \underline{p} \\ \phi(p_{0-})\hat{H}_1(F(p_{0-})) & \text{if} \quad p_{0-} \in (\underline{p}, \bar{p}) \\ V_1^{pip}\left(p_{0-}, \hat{V}^*\right) & \text{if} \quad p_{0-} \ge \bar{p}. \end{cases}$$

Then, $\tilde{V}_1^*(p_{0-})$ is the solution to player 1's transformed problem in Equation (50) and

$$V_1^*(p_{0-}) = \tilde{V}_1^*(p_{0-}) + \frac{\lambda p_{0-} - c}{r+\lambda}$$

is the solution to player 1's problem in Equation (49).

Proof. For $p \in (\underline{p}, \bar{p})$, we have

$$\begin{aligned}\tilde{V}_1^*(p) &= \phi(p)\hat{H}_1(F(p)) \\ &= \phi(p)\frac{L_1(\bar{y}) - G_1(y_1^*)}{\bar{y} - y_1^*}F(p) \\ &= \psi(p)\frac{L_1(\bar{y}) - G_1(y_1^*)}{\bar{y} - y_1^*},\end{aligned}$$

using $F(p) = \psi(p)/\phi(p)$, where $\psi(p)$ is given in Equation (35). Since $\psi(p)$ is a solution to $\mathcal{L}f(p) - (r+\lambda) f(p) = 0$, this implies that

$$\mathcal{L}\tilde{V}_1^*(p) - (r+\lambda)\tilde{V}_1^*(p) = 0$$

for all $p \in (\underline{p}, \bar{p})$. For $p \leq \underline{p}$, we have $\tilde{V}_1^*(p) = \phi(p) G_1(F(p)) = g_1(p)$. Since $g_1(p)$ is a linear function and since $g_1(p) > 0$ for $p \leq \underline{p}$, we have

$$\mathcal{L}\tilde{V}_1^*(p) - (r+\lambda)\tilde{V}_1^*(p) < 0$$

for all $p \leq \underline{p}$. Equation (55) implies that $\hat{H}_1(F(p))$ is differentiable at $\underline{p}$. Thus, $\tilde{V}_1^*(p)$ is differentiable and satisfies $\tilde{V}_1^*(\underline{p}) = g_1(\underline{p})$ and $\tilde{V}_1^{*\prime}(\underline{p}) = g_1'(\underline{p})$. By Lemma 11, the boundary condition (52) holds at $\bar{p}$.

The above results imply that

$$(r+\lambda) V_1^*(p) = \mathcal{L}V_1^*(p) + u_1(p)$$

for all $p \in (\underline{p}, \bar{p})$ and

$$(r+\lambda) V_1^*(p) \geq \mathcal{L}V_1^*(p) + u_1(p)$$

for all $p \leq \underline{p}$. Moreover, we have $V_1^{*\prime}(\underline{p}) = V_1^*(\underline{p}) = 0$ and $V_1^*(\bar{p}) = \frac{1-V_1^*(\bar{p})}{1-\bar{p}}$.

Now, set $V_\xi = 1$, $V_\eta = 0$, $D_t^L = 0$ for all t, and $S = [0, \underline{p}]$ in Proposition 7. Then, Proposition 7 implies that $V_1^*(p_{0-})$ solves player 1's problem in Equation (49). Equivalently, $\tilde{V}_1^*(p_{0-})$ solves player 1's transformed problem in Equation (50). □

The following result follows from the identity (50) and the same argument as in Lemma 5, which characterizes the boundary conditions at $\underline{p}$.

Corollary 3. *$V_1(p_{0-})$ satisfies ODE (11) on $(\underline{p}, \bar{p})$ with boundary conditions $V_1(\underline{p}) = V_1'(\underline{p}) = 0$ and $V_1'(\bar{p}) = \frac{1-V_1(\bar{p})}{1-\bar{p}}$. $V_1(p_{0-})$ is strictly increasing and strictly convex.*

Finally, we have using Equation (55),

$$\frac{dy_1^*}{d\bar{y}} = \frac{L_1''(\bar{y}) + \frac{\partial L_1(\bar{y})}{\partial K}\left(\frac{\partial K}{\partial \hat{V}}\frac{\partial \hat{V}}{\partial \bar{y}} + \frac{\partial K}{\partial \bar{y}}\right)}{G_1''(y_1^*)} > 0 \tag{56}$$

since $\frac{\partial L_1(\bar{y})}{\partial K} > 0$ and since $\frac{dK}{d\bar{y}} = \frac{\partial K}{\partial \hat{V}}\frac{\partial \hat{V}}{\partial \bar{y}} + \frac{\partial K}{\partial \bar{y}} < 0$, i.e., as $\bar{y}$ increases, K needs to decrease for the tangency condition to hold at $\bar{y}$. Note that $y_1^* < y_{max} < \left(\frac{c}{\lambda - c}\right)^A = F\left(\frac{c}{\lambda}\right)$ for any $\bar{y} > F\left(\frac{c}{\lambda}\right)$.

Player 2's Auxiliary Problem. Now, consider player 2's auxiliary problem, so that the sender uses the left-pipetting strategy $D^L(\underline{p})$ and $D^R = 0$. Pick an arbitrary $\hat{V} > 0$ and define for $p \leq \underline{p}$ the pipetting value for player 2 as

$$V_2^{pip}(p, \hat{V}) = \frac{p}{\underline{p}}\hat{V} + \frac{\underline{p} - p}{\underline{p}}.$$

This is the value player 1 gets when the sender randomizes between beliefs $\underline{p}$ and 0 and if the continuation value at $\underline{p}$ is some arbitrary $\hat{V}$. Let

$$V_2(p_{0-}) = \sup_{\tau_2 \in \mathcal{T}} E\left[\int_0^{\tau_1(\underline{p})\wedge\tau_2} e^{-(r+\lambda)t} u_2(p_t)\,dt + e^{-(r+\lambda)\tau_1(\underline{p})} V_2^{pip}\left(\underline{p}, \hat{V}\right) \mathbb{1}\{\tau_2 > \tau_1(\underline{p})\}\right]. \tag{57}$$

This problem is equivalent to

$$\begin{aligned} V_2(p_{0-}) &= \frac{\lambda(1-p_{0-})-c}{r+\lambda} + \sup_{\tau_2\in\mathcal{T}} E\left[e^{-(r+\lambda)\tau_2} g_2(p_{\tau_2}) \mathbb{1}\left\{\tau_2 \le \tau_1(\underline{p})\right\}\right. \\ &\quad \left. + e^{-(r+\lambda)\tau_1(\underline{p})} l_1\left(p_{\tau_1(\underline{p})}\right) \mathbb{1}\left\{\tau_2 > \tau_1(\underline{p})\right\}\right], \end{aligned}$$

where

$$g_2(p) = -\frac{\lambda(1-p)-c}{r+\lambda} \text{ and } l_1(p) = V_2^{pip}(p,\hat{V}) + g_1(p).$$

Define

$$\tilde{V}_2(p_{0-}) = V_2(p_{0-}) - \frac{\lambda(1-p_{0-})-c}{r+\lambda}.$$

Given this transformation, the Robin boundary condition at $\underline{p}$ becomes

$$\begin{aligned} \tilde{V}_2'(\underline{p}) &= \frac{d}{dp} V_2^{pip}(p,\hat{V})\Big|_{p=\underline{p}} + g_2'(\underline{p}) \\ &= \frac{\hat{V}}{\underline{p}} + \frac{\lambda}{r+\lambda}. \end{aligned} \tag{58}$$

As in the no-information benchmark, define $\hat{p}_t = 1 - p_t$ and $\hat{\bar{p}} = 1 - \underline{p}$. Then, we have

$$g_2(1-\hat{p}) = g_1(\hat{p}),$$

$$l_2(1-\hat{p}) = \frac{1-\hat{p}}{1-\hat{\bar{p}}}\hat{V} + \frac{\hat{p}-\hat{\bar{p}}}{1-\hat{\bar{p}}} + g_1(\hat{p}),$$

and

$$V_2^{pip}(1-\hat{p},\hat{V}) = \frac{1-\hat{p}}{1-\hat{\bar{p}}}\hat{V} + \frac{\hat{p}-\hat{\bar{p}}}{1-\hat{\bar{p}}} = V_1^{pip}(\hat{p},\hat{V}).$$

Thus, under this transformation, player 2's problem is identical to player 1's problem, with a given upper threshold $\hat{\bar{p}}$. All arguments from studying player 1's auxiliary problem apply. In particular, define $\hat{y} = F(\hat{p})$, and $\hat{\bar{y}} = F(\hat{\bar{p}})$. Then, there exists a $\hat{V}^*$ and a $\hat{y}_2^*$ so that the Robin boundary condition in Equation (58) holds at $\hat{\bar{p}}$, and player 2 optimally stops at $\hat{y}_2^*$ or equivalently for $p \ge \bar{p} \equiv F^{-1}(\hat{y}_2^*)$. Given $\hat{V}^*$ and the stopping threshold $\hat{y}_2^*$, the analog of the verification argument in Lemma 12 applies. Finally, it again holds that $\frac{d\hat{y}_2^*}{d\hat{\bar{y}}} > 0$ and $\hat{y}_2^* < y_{max} < \left(\frac{c}{\lambda-c}\right)^A$.

Fixed Point. For a given threshold $\bar{p}$, so that the sender uses the right-pipetting strategy $D^R(\bar{p})$, define with $\underline{p}$ the optimal stopping threshold in player 1's auxiliary problem in Equation (49). Given this threshold $\underline{p}$, define with $\bar{P}(\bar{p})$ the optimal stopping threshold in Player 2's auxiliary problem in Equation (57), assuming that the sender uses the left-pipetting strategy $D^L(\underline{p})$. The following Lemma shows that there exists a unique fixed point $\bar{p}^*$, so that $\bar{P}(\bar{p}^*) = \bar{p}^*$.

Lemma 13. *There exists a unique fixed point $\bar{p}^* \in (1 - \frac{c}{\lambda}, 1)$ such that $\bar{P}(\bar{p}^*) = \bar{p}^*$.*

Proof. Fix $\bar{p} \geq \frac{c}{\lambda}$ and define $\bar{y} = F(\bar{p})$. Given $\bar{y}$, player 1 stops at y_1^*. Define $\hat{\bar{y}} = 1/y_1^*$. Then, player 2 stops at $\hat{y}_2^* < \left(\frac{c}{\lambda - c}\right)^A$. Define $\bar{Y} = 1/y_2^*$, and note that $\bar{Y}$ is continuous in $\bar{y}$. Letting $\bar{y} = \left(\frac{c}{\lambda - c}\right)^A$, we have $\bar{Y} > \left(\frac{\lambda - c}{c}\right)^A > \left(\frac{c}{\lambda - c}\right)^A$, and thus $\bar{Y} > \bar{y}$. Letting $\bar{y} \to \infty$ implies that $\hat{y}_2^* \to y_{max}$ and thus $\bar{Y}$ is bounded. Thus, by continuity, the function $\bar{Y}(\bar{y})$ crosses the identity line on the interval $\left[\left(\frac{c}{\lambda - c}\right)^A, \infty\right)$. Hence, a fixed point exists. Since $F(p)$ is a bijection, $\bar{P}(\bar{p})$ admits a fixed point as well.

Suppose that $(\underline{p}, \bar{p})$ is a fixed point. Then, using the transformation $\hat{p} = 1 - p$, we have

$$V_2(1 - p) = V_1(p)$$

and in particular

$$V_2\left(1 - \underline{p}\right) = V_1\left(\underline{p}\right) = 0,$$

since it is optimal for player 1 to stop at $\underline{p}$ given pipetting threshold $\bar{p}$ in Problem (49). In problem (57), it is optimal for player 2 to stop at $\bar{p}$ given pipetting threshold $\underline{p}$, i.e., $V_2(\bar{p}) = 0$. This implies that $\underline{p} + \bar{p} = 1$. Equation (56) implies that $\frac{dy_1^*}{d\bar{y}} > 0$ and thus $\frac{dp}{d\bar{p}} > 0$, so that $\frac{d}{d\bar{p}}\left(\underline{p} + \bar{p}\right) > 1$. Thus, the fixed point is unique. ☐

Sender's Best Response. Define with $\underline{p}^*$ player 1's optimal stopping threshold given $\bar{p}^*$. The following Proposition constructs an MPE using the thresholds $\bar{p}^*$ and $\underline{p}^*$, so that the pipetting strategy $\{D^L(\underline{p}^*), D^R(\bar{p}^*)\}$ is a best response to the stopping times $\tau_1(\underline{p}^*)$ and $\tau_2(\bar{p}^*)$.

Proposition 8. *Fix $\underline{p}$ and $\bar{p}$ with $0 < \underline{p} < \bar{p} < 1$. The pipetting strategy $\{D^L(\underline{p}), D^R(\bar{p})\}$ is the essentially unique best response for the sender to the stopping strategies*

$$\tau(\underline{p}) = \inf\left\{t \geq 0 : p_t < \underline{p}\right\} \text{ and } \tau(\bar{p}) = \inf\left\{t \geq 0 : p_t > \bar{p}\right\}.$$

To establish Proposition 8, I first formulate a verification theorem for general disclosure strategies $D \in \mathcal{D}$. Then, I construct a candidate best response, and use the verification theorem to prove that it is indeed optimal for the sender.

Proposition 9. *For an arbitrary stopping time τ, define*

$$W(p_{0-}) = \sup_{D \in \mathcal{D}} E\left[\int_0^{\tau \wedge \tau_\xi \wedge \tau_\eta} e^{-(r+\lambda)t} w dt\right].$$

Suppose that there exists a function $f(p)$ which is continuously differentiable for all p and twice continuously differentiable for almost all p and which satisfies

(i) $f(p) + (1 - p) f'(p) \geq 0$ for all p,

(ii) $f(p) - p f'(p) \geq 0$ for all p,

(iii) for all $l^R, l^L \in [0,1]$ *and* $p \in [0,1]$,

$$f(p) \geq \left(pl^R + (1-p)\, l^L\right) f\left(\frac{pl^R}{pl^R + (1-p)\, l^L}\right),$$

(iv) $(r+\lambda) f(p) = \mathcal{L} f(p) + w$ *for all* $p \in \mathcal{N}$, *where the non-disclosure region* $\mathcal{N}$ *is given by* $\mathcal{N} = \{p : \text{(i), (ii), and (iii) are strict}\}$.

Moreover, suppose that there exists a disclosure strategy $\hat{D} \in \mathcal{D}$ *such that*

(v) $d\hat{D}_t^L = d\hat{D}_t^R = 0$ *for all* $p \in \mathcal{N}$,

(vi) $(f(p) + (1-p) f'(p))\, d\hat{D}_t^{R,c} = 0$ *for all* p,

(vii) $(f(p) - pf'(p))\, d\hat{D}_t^{L,c} = 0$ *for all* p,

(viii) for all times t *at which* $\hat{D}_t^L \neq \hat{D}_{t-}^L$ *or* $\hat{D}_t^R \neq \hat{D}_{t-}^R$, *we have*

$$f(p_{t-}) = \left(p_{t-} l_t^R + (1-p_{t-})\, l_t^L\right) f\left(\frac{p_{t-} l_t^R}{p_{t-} l_t^R + (1-p_{t-})\, l_t^L}\right),$$

where $l_t^R = \frac{1-\hat{D}_t^R}{1-\hat{D}_{t-}^R}$ *and* $l_t^L = \frac{1-\hat{D}_t^L}{1-\hat{D}_{t-}^L}$.

Then, $f(p_{0-}) = W(p_{0-})$ *and the disclosure strategy* $\hat{D}$ *is optimal.*

Proof. Consider an arbitrary disclosure strategy $D \in \mathcal{D}$. As in the proof of Proposition 7, for any almost surely finite stopping time τ,

$$E\left[\int_0^{\tau \wedge \tau_\eta \wedge \tau_\xi} e^{-(r+\lambda)t} dt\right] \quad = \quad E\left[\int_0^{\tau} e^{-(r+\lambda)t} \pi(D_t)\, dt\right],$$

where again $\pi(D_t) = p_{0-}\left(1 - D_t^R\right) + (1-p_{0-})\left(1 - D_t^L\right)$. Thus,

$$W(p_{0-}) \quad = \quad \sup_{D \in \mathcal{D}} wE\left[\int_0^{\tau} e^{-(r+\lambda)t} \pi(D_t)\, dt\right].$$

As in the proof of Proposition 7, applying Harrison (2013), Prop. 4.14, p. 70, and Ito's Lemma for semi-martingales (Protter (2005), Th. II.32, p. 78) to the process $\tilde{f}_t \equiv e^{-(r+\lambda)t} \pi(D_t) f(p_t)$ together with a localization argument yields

$$\begin{aligned}
f(p_{0-}) \quad = \quad & E\left[\int_0^{\tau} e^{-(r+\lambda)t} \pi(D_t)\left((r+\lambda) f(p_t) - \mathcal{L} f(p_t)\right) dt\right] \\
& + E\left[\int_0^{\tau} e^{-(r+\lambda)t}\left(p_{0-} f(p_t) + f'(p_t)\, \pi(D_t) \frac{p_t(1-p_t)}{1-D_t^R}\right) dD_t^{R,c}\right] \\
& + E\left[\int_0^{\tau} e^{-(r+\lambda)t}\left((1-p_{0-}) f(p_t) - f'(p_t)\, \pi(D_t) \frac{p_t(1-p_t)}{1-D_t^L}\right) dD_t^{L,c}\right] \\
& + E\left[\sum_{0 \leq s \leq t} \left(\pi(D_{s-}) f(p_{s-}) - \pi(D_s) f(p_s)\right)\right].
\end{aligned}$$

The same argument as in the proof of Proposition 7 (in particular Equation (48)) implies that

$$p_{0-} f(p_t) + f'(p_t)\,\pi(D_t)\,\frac{p_t(1-p_t)}{1-D_t^R} \quad = \quad p_{0-}\left(f(p_t) + f'(p_t)(1-p_t)\right).$$

Then (i) implies that

$$p_{0-}\left(f(p_t) + f'(p_t)(1-p_t)\right) \geq 0.$$

Similarly, Equation (48) and (ii) imply that

$$(1-p_{0-})\,f(p_t) - f'(p_t)\,\pi(D_t)\,\frac{p_t(1-p_t)}{1-D_t^L} \quad = \quad (1-p_{0-})\left(f(p_t) - f'(p_t)\,p_t\right) \geq 0.$$

Finally, for any time t at which $p_t \neq p_{t-}$, the same argument as in Equation (48) implies that

$$p_{0-} = p_{0-} D_{t-}^R + p_{t-}\pi(D_{t-}),$$

so that

$$\pi(D_t) = \frac{p_{0-}\left(1-D_t^R\right)}{p_t} \text{ and } \pi(D_{t-}) = \frac{p_{0-}\left(1-D_{t-}^R\right)}{p_{t-}}.$$

Bayes' rule implies that

$$p_t = \frac{p_{t-} l_t^R}{p_{t-} l_t^R + (1-p_{t-})\, l_t^L},$$

where $l_t^R = \frac{1-D_t^R}{1-D_{t-}^R}$ and $l_t^L = \frac{1-D_t^L}{1-D_{t-}^L}$. Note that $l_t^R, l_t^L \in [0,1]$. Then,

$$\frac{\pi(D_t)}{\pi(D_{t-})} = p_{t-} l_t^R + (1-p_{t-})\, l_t^L \tag{59}$$

and hence

$$\begin{aligned}
\pi(D_{t-}) f(p_{t-}) - \pi(D_t) f(p_t) \quad &= \quad \pi(D_{t-})\left(f(p_{t-}) - \frac{\pi(D_t)}{\pi(D_{t-})} f(p_t)\right) \\
&= \quad \pi(D_{t-})\left(f(p_{t-}) - \left(p_{t-} l_t^R + (1-p_{t-})\, l_t^L\right) f\left(\frac{p_{t-} l_t^R}{p_{t-} l_t^R + (1-p_{t-})\, l_t^L}\right)\right) \\
&\geq \quad 0
\end{aligned}$$

by (iii). Collecting equations and using (iv) implies that

$$f(p_{0-}) \quad \geq \quad wE\left[\int_0^\tau e^{-(r+\lambda)t}\pi(D_t)\,dt\right].$$

Since the strategy D was arbitrary, we have

$$f(p_{0-}) \geq \sup_{D\in\mathcal{D}} wE\left[\int_0^\tau e^{-(r+\lambda)t}\pi(D_t)\,dt\right].$$

Now, picking $D = \hat{D}$, the same argument as above together with conditions (v)–(viii) yields

$$f(p_{0-}) = wE\left[\int_0^\tau e^{-(r+\lambda)t}\pi\left(\hat{D}_t\right)dt\right].$$

Thus, $f(p_{0-}) = W(p_{0-})$ and $\hat{D}$ is optimal. $\square$

Condition (iii) in the above proposition is difficult to verify for a particular candidate value function. The following lemma provides simpler sufficient conditions.

Lemma 14. *If*

$$f(p) \geq \frac{1-p}{1-p'} f(p') \tag{60}$$

for all $p' \leq p$ *and*

$$f(p) \geq f(p')\frac{p}{p'} \tag{61}$$

for all $p' \geq p$*, then for all* $l^R, l^L \in [0,1]$ *and* $p \in [0,1]$*,*

$$f(p) \geq \left(pl^R + (1-p)l^L\right) f\left(\frac{pl^R}{pl^R + (1-p)l^L}\right). \tag{62}$$

If Inequality (60) is strict for all $p' < p$ *and Inequality (61) is strict for all* $p' > p$*, then Inequality (62) is strict for all* l^R *and* l^L *such that* $l^R \cdot l^L < 1$*.*

Proof. Suppose that $l^R < l^L \leq 1$. Then,

$$\begin{aligned}
\left(pl^R + (1-p)l^L\right) f\left(\frac{pl^R}{pl^R + (1-p)l^L}\right) &= l^L\left(p\frac{l^R}{l^L} + (1-p)\right) f\left(\frac{p\frac{l^R}{l^L}}{p\frac{l^R}{l^L} + 1 - p}\right) \\
&< \left(p\tilde{l}^R + (1-p)\right) f\left(\frac{p\tilde{l}^R}{p\tilde{l}^R + 1 - p}\right),
\end{aligned}$$

where $\tilde{l}^R = l^R/l^L < 1$. Define

$$p' = \frac{p\tilde{l}^R}{p\tilde{l}^R + 1 - p}$$

and note that $p' < p$ since $\tilde{l}^R < 1$. We have

$$p\tilde{l}^R + (1-p) = \frac{1-p}{1-p'},$$

which follows from algebra. Thus,

$$\left(p\tilde{l}^R + (1-p)\right) f\left(\frac{p\tilde{l}^R}{p\tilde{l}^R + 1 - p}\right) = \frac{1-p}{1-p'} f(p') \leq f(p),$$

which implies that Inequality (62) holds strictly.

Suppose now that $l^L < l^R \leq 1$. Then an analogous calculation yields

$$\left(pl^R + (1-p)\,l^L\right) f\left(\frac{pl^R}{pl^R + (1-p)\,l^L}\right) < \left(p + (1-p)\,\tilde{l}^L\right) f\left(\frac{p}{p + (1-p)\,\tilde{l}^L}\right)$$

where $\tilde{l}^L = l^L / l^R < 1$. Define

$$p' = \frac{p}{p + (1-p)\,\tilde{l}^L}$$

and note that $p' > p$. We have

$$p + (1-p)\,\tilde{l}^L = \frac{p}{p'},$$

which again follows from algebra. Thus,

$$\left(p\tilde{l}^R + (1-p)\right) f\left(\frac{p\tilde{l}^R}{p\tilde{l}^R + 1 - p}\right) = \frac{p}{p'} f\left(p'\right) \leq f\left(p\right),$$

which implies that Inequality (62) holds strictly. Finally, if $l^L = l^R$, then trivially

$$\left(pl^R + (1-p)\,l^L\right) f\left(\frac{pl^R}{pl^R + (1-p)\,l^L}\right) = \left(pl^R + (1-p)\,l^L\right) f\left(p\right) \leq f\left(p\right),$$

and the inequality is strict whenever $l^L \cdot l^R < 1$. □

Now, consider the function $\hat{W}(p)$, which is defined as follows. For $p \in (\underline{p}, \bar{p})$, $\hat{W}(p)$ satisfies the ODE (13) with boundary conditions

$$\hat{W}(\underline{p}) - \underline{p}\hat{W}'(\underline{p}) = 0 \text{ and } \hat{W}(\bar{p}) + (1-\bar{p})\,\hat{W}'(\bar{p}) = 0,$$

and

$$\hat{W}(p) = \begin{cases} \frac{1-p}{1-\bar{p}}\hat{W}(\bar{p}) & \text{for} \quad p > \bar{p} \\ \frac{p}{\underline{p}}\hat{W}(\underline{p}) & \text{for} \quad p < \underline{p}. \end{cases}$$

The Lemma below verifies that such a solution to the ODE indeed exists and is unique.

Lemma 15. *Fix two thresholds $0 < \underline{p} < \bar{p} < 1$. Then, the ODE (13) with boundary conditions*

$$W\left(\underline{p}\right) - \underline{p}W'\left(\underline{p}\right) = 0 \text{ and } W\left(\bar{p}\right) + (1-\bar{p})\,W'\left(\bar{p}\right) = 0 \tag{63}$$

has a unique twice continuously differentiable solution.

Proof. The proof is analogous to the one in Lemma 9 and uses the method of lower and upper solutions (De Coster and Habets (2006), Th. 1.5, p. 81). A twice continuously differentiable function $\alpha(p)$ is an upper solution if

$$\alpha''(p) \geq \frac{2}{\sigma(p)^2}\left((r+\lambda)\,\alpha(p) - w\right)$$

for $p \in (\underline{p}, \bar{p})$, and $\alpha(\underline{p}) - \underline{p}\alpha'(\underline{p}) \leq 0$ and $\alpha(\bar{p}) + (1-\bar{p})\alpha'(\bar{p}) \leq 0$. Picking $\alpha(p) = -Me^p$ as in the proof of Lemma 9 implies that $\alpha(p)$ is a lower solution. Similarly, an upper solution is a twice continuously differentiable function $\beta(p)$ such that

$$\beta''(p) \leq \frac{2}{\sigma(p)^2}\left((r+\lambda)\beta(p) - w\right)$$

for $p \in (\underline{p}, \bar{p})$, and $\beta(\underline{p}) - \underline{p}\beta'(\underline{p}) \geq 0$ and $\beta(\bar{p}) + (1-\bar{p})\beta(\bar{p}) \geq 0$. Trivially, $\beta(p) = 0$ is an upper solution. Now, De Coster and Habets (2006), Th. 1.5, p. 81, establishes that ODE (13) with boundary conditions (63) has a solution. Uniqueness follows from the same comparison argument as in the proof of Lemma 9. $\square$

Now, applying Proposition 7 with $V_\xi = V_\eta = 0$ to the function $\hat{W}(p)$ implies that $\hat{W}(p)$ equals the sender's value function given the pipetting strategy $\{D^L(\underline{p}), D^R(\bar{p})\}$. This implies that $\hat{W}(p) < \frac{w}{r+\lambda}$ for all $p \in [0,1]$, where $\frac{w}{r+\lambda}$ is the sender's value when both players continue forever.

Lemma 16. *$\hat{W}(p)$ is strictly concave for $p \in (\underline{p}, \bar{p})$.*

Proof. We have

$$\hat{W}''(p)\frac{1}{2}\sigma(p)^2 = (r+\lambda)\hat{W}(p) - w < 0,$$

since $\hat{W}(p) < \frac{w}{r+\lambda}$. $\square$

It remains to establish that $\hat{W}(p)$ satisfies all conditions in the statement of Proposition 9. This immediately implies that $\hat{W}(p)$ is the sender's optimal value function given the stopping strategies $\tau_1(\underline{p})$ and $\tau_2(\bar{p})$, and that the pipetting strategy $\{D^L(\underline{p}), D^R(\bar{p})\}$ is the sender's best response.

Lemma 17. *$\hat{W}(p)$ satisfies conditions (i)–(viii) in Proposition 9.*

Proof. (i): For $p \leq \underline{p}$, $\hat{W}(p) = \frac{p}{\underline{p}}\hat{W}(\underline{p}) > 0$ and $\hat{W}'(p) = \frac{1}{\underline{p}}\hat{W}(\underline{p})$, which implies that $\hat{W}(p) + (1-p)\hat{W}'(p) > 0$. For $p \geq \bar{p}$, we have $\hat{W}(p) = \frac{1-p}{1-\bar{p}}\hat{W}(\bar{p})$ and $\hat{W}'(p) = -\frac{1}{1-\bar{p}}\hat{W}(\bar{p})$, so that $\hat{W}(p) + (1-p)\hat{W}'(p) = 0$. We have $\hat{W}(\bar{p}) + (1-\bar{p})\hat{W}'(\bar{p}) = 0$, and since $\hat{W}(p)$ is strictly concave on $(\underline{p}, \bar{p})$ we have $\hat{W}(p) + (1-p)\hat{W}'(p) > 0$ for all $p \in (\underline{p}, \bar{p})$. Thus, $\hat{W}(p) + (1-p)\hat{W}'(p) \geq 0$ for all p.

(ii): For $p \leq \underline{p}$, $\hat{W}(p) - p\hat{W}'(p) = 0$. For $p \geq \bar{p}$,

$$\hat{W}(p) - p\hat{W}'(p) = \frac{1-p}{1-\bar{p}}\hat{W}(\bar{p}) + \frac{p}{1-\bar{p}}\hat{W}(\bar{p}) = \frac{1}{1-\bar{p}}\hat{W}(\bar{p}) > 0.$$

We have $\hat{W}(\underline{p}) - \underline{p}\hat{W}'(\underline{p}) = 0$, and since $\hat{W}(p)$ is strictly concave on $(\underline{p}, \bar{p})$, we have $\hat{W}(p) - p\hat{W}'(p) > 0$ for all $p \in (\underline{p}, \bar{p})$. Thus, $\hat{W}(p) - p\hat{W}'(p) \geq 0$ for all p.

(iii): The argument uses the sufficient conditions in Lemma 9. It is divided into two parts:

(iii-a): Consider the inequality $\hat{W}(p) \geq \frac{1-p}{1-p'}\hat{W}(p')$ for $p' \leq p$. For $p, p' \in [\bar{p}, 1]$, the inequality binds, since $\hat{W}(p) = \frac{1-p}{1-\bar{p}}\hat{W}(\bar{p})$ for any such p and p'. Now, pick $p \geq \bar{p}$ and $p' \in (\underline{p}, \bar{p})$. Since $\hat{W}(p)$ is strictly concave for $p \in (\underline{p}, \bar{p})$, it holds that $\hat{W}'(p) > \hat{W}'(\bar{p}) = -\frac{1}{1-\bar{p}}\hat{W}(\bar{p})$ for any such p. This implies that $\hat{W}(p') < \frac{1-p'}{1-\bar{p}}\hat{W}(\bar{p})$. But since $\frac{1-p'}{1-\bar{p}}\hat{W}(\bar{p}) = \frac{1-p'}{1-p}\hat{W}(p)$,we have $\hat{W}(p') < \frac{1-p'}{1-p}\hat{W}(p)$ or equivalently $\hat{W}(p) > \frac{1-p}{1-p'}\hat{W}(p')$. For $p' \leq \underline{p}$, $\hat{W}(p')$ is strictly increasing in p' while the line

$\frac{1-p'}{1-\bar{p}}\hat{W}(\bar{p})$ is strictly decreasing. Since $\hat{W}(\underline{p}) < \frac{1-\underline{p}}{1-\bar{p}}\hat{W}(\bar{p})$, we have $\hat{W}(p') < \frac{1-p'}{1-\bar{p}}\hat{W}(\bar{p})$ for any $p' \leq \underline{p}$, which again implies that $\hat{W}(p) > \frac{1-p}{1-p'}\hat{W}(p')$. Next, pick $p \in (\underline{p}, \bar{p})$, and pick $p' \in (\underline{p}, \bar{p})$ with $p' < p$. Since $\hat{W}(p)$ is strictly concave for $p \in (\underline{p}, \bar{p})$, and since $\hat{W}(\bar{p}) + (1-\bar{p})\hat{W}'(\bar{p}) = 0$, we have $\hat{W}'(p) > -\frac{1}{1-p}\hat{W}(p)$. Together with concavity, this implies that $\hat{W}(p') < \frac{1-p'}{1-p}\hat{W}(p)$, or equivalently $\hat{W}(p) > \frac{1-p}{1-p'}\hat{W}(p')$. For $p \in (\underline{p}, \bar{p})$ and $p' \leq \underline{p}$, we again have $\hat{W}(p') < \frac{1-p'}{1-p}\hat{W}(p)$, because $\hat{W}(p') \leq \hat{W}(\underline{p}) < \frac{1-\underline{p}}{1-p}\hat{W}(p)$, since $\hat{W}(p')$ is strictly increasing in p' for $p' \leq \underline{p}$, while $\frac{1-p'}{1-p}W(p)$ is strictly decreasing in p'. Finally, for $p \leq \underline{p}$ and $p' < p$, we have $\hat{W}(p') < \frac{1-p'}{1-p}\hat{W}(p)$ because $\hat{W}(p')$ is strictly increasing in p' for $p' \leq \underline{p}$, while $\frac{1-p'}{1-p}W(p)$ is strictly decreasing in p'. The preceding argument establishes that $\hat{W}(p) \geq \frac{1-p}{1-p'}\hat{W}(p')$ for $p' \leq p$.

(iii-b): Consider the inequality $\hat{W}(p) \geq \hat{W}(p')\frac{p}{p'}$ for $p' \geq p$. The argument is analogous to the previous case. Picking $p, p' \leq \underline{p}$ implies that $\hat{W}(p) = \frac{p}{\underline{p}}\hat{W}(\underline{p})$ and $\hat{W}(p') = \frac{p'}{\underline{p}}\hat{W}(\underline{p})$ and hence $\hat{W}(p) = \frac{p}{p'}\hat{W}(p')$. Now, pick $p \leq \underline{p}$ and $p' \in (\underline{p}, \bar{p})$. Since $\hat{W}'(\underline{p}) = \frac{1}{\underline{p}}\hat{W}(\underline{p})$ and since $\hat{W}(p)$ is strictly concave for $p \in (\underline{p}, \bar{p})$, we have $\hat{W}(p) < \frac{1}{\underline{p}}\hat{W}(\underline{p})$ for all $p \in (\underline{p}, \bar{p})$. This implies that $\hat{W}(p') < \frac{p'}{\underline{p}}\hat{W}(\underline{p})$ and thus $\hat{W}(p') < \frac{p'}{p}\hat{W}(p)$ or equivalently $\hat{W}(p) > \frac{p}{p'}\hat{W}(p')$. Next, pick $p \leq \underline{p}$ and $p' \geq \bar{p}$. Then, the fact that $\hat{W}(\bar{p}) < \frac{\bar{p}}{\underline{p}}\hat{W}(\underline{p})$ and the fact that $\hat{W}(p')$ is strictly decreasing for $p' \geq \bar{p}$ and $\frac{p'}{\underline{p}}\hat{W}(\underline{p})$ is strictly increasing implies that $\hat{W}(p') < \frac{p'}{p}W(p)$. Now, pick $p, p' \in (\underline{p}, \bar{p})$ with $p' > p$. Since $\hat{W}(p)$ is strictly concave on $(\underline{p}, \bar{p})$, and since $\hat{W}'(\underline{p}) = \frac{1}{\underline{p}}\hat{W}(\underline{p})$, we have $\hat{W}'(p) < \frac{1}{p}\hat{W}(p)$, and for all $\tilde{p} \in (p, p')$, $\hat{W}'(\tilde{p}) < \frac{1}{p}\hat{W}(p)$. Thus, $\hat{W}(p') < \frac{p'}{p}\hat{W}(p)$. Next, pick $p \in (\underline{p}, \bar{p})$ and $p' \geq \bar{p}$. Then, we again have $\hat{W}(p') < \frac{p'}{p}\hat{W}(p)$, because $\hat{W}(\bar{p}) < \frac{\bar{p}}{\underline{p}}\hat{W}(\underline{p})$, $\hat{W}(p')$ is strictly decreasing for $p' \geq \bar{p}$, and $\frac{p'}{\underline{p}}\hat{W}(\underline{p})$ is strictly increasing. Finally, pick $p, p' \geq \bar{p}$. Then, since $\hat{W}(p)$ is strictly decreasing for $p \geq \bar{p}$, we immediately have $\hat{W}(p') < \frac{p'}{p}W(p)$. Overall, the preceding argument establishes that $\hat{W}(p) \geq \hat{W}(p')\frac{p}{p'}$ for $p' \geq p$.

(iv): This follows immediately from Lemma 15, defining the no-disclosure region as $\mathcal{N} = (\underline{p}, \bar{p})$.

(v): This holds by construction, since given the pipetting strategy $\{D^L(\underline{p}), D^R(\bar{p})\}$, the sender discloses no information for $p \in \mathcal{N}$.

(vi): Under the pipetting strategy $\{D^L(\underline{p}), D^R(\bar{p})\}$, we have $dD_t^{R,c} > 0$ only if $p_t = \bar{p}$. Then, the boundary condition $\hat{W}(\bar{p}) + (1-\bar{p})\hat{W}'(\bar{p})$ implies that (vi) holds.

(vii): Similarly, we have $dD_t^{L,c}$ only if $p_t = \underline{p}$ and the boundary condition $\hat{W}(\underline{p}) - \underline{p}\hat{W}'(\underline{p}) = 0$ implies that (vii) holds.

(viii): Given the pipetting strategy, $\{D^L(\underline{p}), D^R(\bar{p})\}$, we have $D_t^R > D_{t-}^R$ if and only if $t = 0-$ and $p_{0-} < \bar{p}$, in which case $D_0^L = D_{0-}^L = 0$ and $\hat{W}(p_{0-}) = \frac{1-p_{0-}}{1-\bar{p}}\hat{W}(\bar{p})$. Then, $l_0^R = 1 - D_0^R$ and $l_0^L = 1$, so that

$$\left(p_{0-}l_0^R + (1-p_{0-})\,l_0^L\right)\hat{W}\left(\frac{p_{0-}l_0^R}{p_{0-}l_0^R + (1-p_{0-})\,l_0^L}\right) = \left(p_{0-}l_0^R + 1 - p_{0-}\right)\hat{W}\left(\frac{p_{0-}l_0^R}{p_{0-}l_0^R + 1 - p_{0-}}\right).$$

Using Equation (3) implies that

$$\frac{p_{0-}l_0^R}{p_{0-}l_0^R + 1 - p_{0-}} = \bar{p}$$

and after some algebra, it follows that

$$p_{0-}l_0^R + 1 - p_{0-} = \frac{1 - p_{0-}}{1 - \bar{p}}.$$

Thus,

$$\hat{W}\left(p_{0-}\right) = \frac{1 - p_{0-}}{1 - \bar{p}}\hat{W}(\bar{p}) = \left(p_{0-}l_0^R + \left(1 - p_{0-}\right)l_0^L\right)\hat{W}\left(\frac{p_{0-}l_0^R}{p_{0-}l_0^R + \left(1 - p_{0-}\right)l_0^L}\right).$$

Similarly, we have $D_t^L > D_{t-}^L$ if and only if $t = 0-$ and $p_{0-} < \underline{p}$, in which case $D_0^R = D_{0-}^R = 0$ and $\hat{W}\left(p\right) = \frac{p}{\underline{p}}\hat{W}(\underline{p})$. Then, $l_0^L = 1 - D_0^L$ and $l_0^R = 1$, so that

$$\left(p_{0-}l_0^R + \left(1 - p_{0-}\right)l_0^L\right)\hat{W}\left(\frac{p_{0-}l_0^R}{p_{0-}l_0^R + \left(1 - p_{0-}\right)l_0^L}\right) = \left(p_{0-} + \left(1 - p_{0-}\right)l_0^L\right)\hat{W}\left(\frac{p_{0-}}{p_{0-} + \left(1 - p_{0-}\right)l_0^L}\right).$$

Equation (5) implies that

$$\frac{p_{0-}}{p_{0-} + \left(1 - p_{0-}\right)l_0^L} = \underline{p}$$

and after some algebra, it follows that

$$p_{0-} + \left(1 - p_{0-}\right)l_0^L = \frac{p}{\underline{p}}.$$

Hence,

$$\hat{W}\left(p\right) = \frac{p}{\underline{p}}\hat{W}(\underline{p}) = \left(p_{0-}l_0^R + \left(1 - p_{0-}\right)l_0^L\right)\hat{W}\left(\frac{p_{0-}l_0^R}{p_{0-}l_0^R + \left(1 - p_{0-}\right)l_0^L}\right).$$

Overall, the function $\hat{W}(p)$ satisfies conditions (i)–(viii) in Proposition 9. □

Existence of MPE. The previous results have established that if players 1 and 2 use stopping strategies $\tau_1(\underline{p}^*)$ and $\tau_2(\bar{p}^*)$, then it is optimal for the sender to use the pipetting strategies $\{D^L(\underline{p}^*), D^R(\bar{p}^*)\}$. The following result establishes the converse. Hence the tuple $\{D^L(\underline{p}^*), D^R(\bar{p}^*), \tau_1(\underline{p}^*), \tau_2(\bar{p}^*)\}$ constitutes an MPE.

Proposition 10. *The stopping strategies $\tau_1(\underline{p}^*)$ and $\tau_2(\bar{p}^*)$ are the essentially unique best responses to the pipetting strategy $\{D^L(\underline{p}^*), D^R(\bar{p}^*)\}$ and to each other.*

Proof. Consider player 1. Lemma 12 establishes that $\tau_1(\underline{p}^*)$ is the best response to the strategies $D^R(\bar{p}^*)$ and $D^L = 0$. By Lemma 12, we have $V_1(\underline{p}^*) = V_1'(\underline{p}^*) = 0$ and $V_1(p) = 0$ for all $p < \underline{p}^*$. Given the pipetting strategy $D^L(\underline{p}^*)$, the verification argument in Proposition 7 implies that player 1's value must satisfy the Robin boundary condition

$$V_1'(\underline{p}^*) = \frac{V_1(\underline{p}^*)}{\underline{p}^*}.$$

But since $V_1(\underline{p}^*) = V_1'(\underline{p}^*) = 0$, this condition is trivially satisfied. Thus, Proposition 7 immediately implies that the solution to player 1's auxiliary problem in Equation (49) is also the solution under the pipetting strategy $\{D^L(\underline{p}^*), D^R(\bar{p}^*)\}$. Thus, $\tau_1^*(\underline{p}^*)$ is optimal for player 1. The argument for player 2 is analogous. Lemma 13 implies that given the pipetting strategy $\{D^L(\underline{p}^*), D^R(\bar{p}^*)\}$, the stopping strategies $\tau_1^*(\underline{p}^*)$ and $\tau_2^*(\bar{p}^*)$ are indeed optimal for both players. $\square$

Uniqueness of MPE. The following arguments establish that the equilibrium constructed in Proposition 2 is unique. First, note that in any MPE we have $S_1 \subset [0, \frac{c}{\lambda})$ and $S_2 \subset (1 - \frac{c}{\lambda}, 1]$, so that $S_1 \cap S_2 = \emptyset$ and $(0,1) \setminus (S_1 \cup S_2) \neq \emptyset$. This implies that there exists an interval $(\underline{p}, \bar{p})$ such that neither player stops for $p \in (\underline{p}, \bar{p})$. It remains to show that (1) this interval is unique, i.e. there is no other interval on which both players choose to continue; (2) the sender uses the pipetting strategies $\{D^L(\underline{p}), D^R(\bar{p})\}$, and in particular discloses no information for $p \in (\underline{p}, \bar{p})$, and (3) $\underline{p} = \underline{p}^*$ and $\bar{p} = \bar{p}^*$.

I first establish that for any interval $(\underline{p}, \bar{p})$ so that both players continue, the sender discloses no information. The following auxiliary result simplifies the argument.

Lemma 18. *For any optimal policy $D \in \mathcal{D}$ and $t < \tau$, $D_t^R > D_{t-}^R$ if and only if $D_t^L = D_{t-}^L$.*

Proof. Fix an optimal policy D and a time $t < \tau$ so that $D_t^R > D_{t-}^R$ and $D_t^L > D_{t-}^L$. Define $l_t^R = \frac{1-D_t^R}{1-D_{t-}^R}$ and $l_t^L = \frac{1-D_t^L}{1-D_{t-}^L}$. Let p_t denote the posterior belief given prior p_{t-}. Bayes rule implies that

$$p_t = \frac{p_{t-} l_t^R}{p_{t-} l_t^R + (1 - p_{t-})\, l_t^L}.$$

If $l_t^R < l_t^L < 1$, one can induce the same posterior p_t by setting $\tilde{l}_t^R = l_t^R / l_t^L$ and $\tilde{l}_t^L = 1$. We have

$$\Pr\left(t > \tau_\xi \wedge \tau_\eta | t- > \tau_\xi \wedge \tau_\eta\right) = \frac{\pi\left(D_t\right)}{\pi\left(D_{t-}\right)}$$

and the same argument as in the proof of Proposition 9 (see Equation (59)) implies that

$$\begin{aligned} \frac{\pi\left(D_t\right)}{\pi\left(D_{t-}\right)} &= p_{t-}\left(1 - D_{t-}^R\right) l_t^R + \left(1 - p_{t-}\right)\left(1 - D_{t-}^L\right) l_t^L \\ &= l_t^L \left(p_{t-}\left(1 - D_{t-}^R\right) \tilde{l}_t^R + \left(1 - p_{t-}\right)\left(1 - D_{t-}^L\right)\right) \\ &< \tilde{l}_t^L \left(p_{t-}\left(1 - D_{t-}^R\right) \tilde{l}_t^R + \left(1 - p_{t-}\right)\left(1 - D_{t-}^L\right)\right). \end{aligned}$$

Thus, given $\tilde{l}_t^R$ and $\tilde{l}_t^L$, the posterior belief is the same, but the probability that $t > \tau_\xi \wedge \tau_\eta$ strictly increases, so using $\tilde{l}_t^R$ and $\tilde{l}_{\tau_D}^L$ is a strict improvement for the sender. The cases $l_{\tau_D}^R < l_{\tau_D}^L < 1$ and $l_{\tau_D}^R = l_{\tau_D}^L < 1$ are analogous. $\square$

Lemma 19. *For any interval $(\underline{p}, \bar{p})$ so that both players continue, no disclosure is optimal for the sender without loss of generality.*

Proof. Consider an arbitrary disclosure strategy D. Define $\tau_D = \inf\left\{t \geq 0 : p_t \neq (\underline{p}, \bar{p})\right\}$. The sender's continuation value at time $t < \tau_D$ is given by

$$\begin{aligned} &E\left[w \int_t^{\tau_D \wedge \tau_\xi \wedge \tau_\eta} e^{-(r+\lambda)(s-t)} ds + e^{-(r+\lambda)(\tau_D - t)} W\left(p_{\tau_D}\right) \mathbb{1}\{\tau_D < \tau_\xi \wedge \tau_\eta\}\right] \\ &= E\left[w \int_t^{\tau_D} e^{-(r+\lambda)(s-t)} \frac{\pi\left(D_s\right)}{\pi\left(D_t\right)} ds + e^{-(r+\lambda)(\tau_D - t)} W\left(p_{\tau_D}\right) \frac{\pi\left(D_{\tau_D}\right)}{\pi\left(D_t\right)}\right], \end{aligned}$$

using Equation (46). We have

$$E\left[w\int_t^{\tau_D} e^{-(r+\lambda)(s-t)}\frac{\pi(D_s)}{\pi(D_t)}ds+e^{-r\tau_D}W(p_{\tau_D})\frac{\pi(D_{\tau_D})}{\pi(D_t)}\right]$$
$$\leq E\left[w\int_t^{\tau_D} e^{-(r+\lambda)(s-t)}ds+e^{-r\tau_D}W(p_{\tau_D})\frac{\pi(D_{\tau_D})}{\pi(D_t)}\right].$$

The upper bound on the RHS can be achieved by fixing the stopping time τ_D and by picking a disclosure strategy $\hat{D}$ so that $\hat{D}_s^R = D_t^R$ and $\hat{D}_s^L = D_t^L$ for $s \in (t, \tau_D)$, and $\hat{D}_{\tau_D}^R = D_{\tau_D}^R$ and $\hat{D}_{\tau_D}^L = D_{\tau_D}^L$. That is, strategy $\hat{D}$ provides no disclosure for $s \in (t, \tau_D)$ and then discloses with the same cumulative probability as strategy D for $t = \tau_D$.

The above argument implies that any strategy in which information is disclosed continuously (i.e., $dD_t^{R,c} > 0$ or $dD_t^{L,c} > 0$) on $(\underline{p}, \bar{p})$ is suboptimal. It remains to show that any strategy in which the belief jumps at τ_D is suboptimal as well. To this end, define with $p_{\tau_D-} \in (\underline{p}, \bar{p})$ the belief right before the jump, pick an $\varepsilon > 0$ so that $(p_{\tau_D-} - \varepsilon, p_{\tau_D-} + \varepsilon) \subset (\underline{p}, \bar{p})$, and define $\tau(p_{\tau_D-} + \varepsilon) = \inf\{t \geq \tau_D : p_t = p_{\tau_D-} + \varepsilon\}$ and $\tau(p_{\tau_D-} - \varepsilon) = \inf\{t \geq \tau_D : p_t = p_{\tau_D-} - \varepsilon\}$. Consider the following disclosure strategy $\hat{D}$, which satisfies $\hat{D}_t^R = D_{\tau_D-}^R$ and $\hat{D}_t^L = D_{\tau_D-}^L$ for all $t \in (\tau_D, \tau(p_{\tau_D-} + \varepsilon) \wedge \tau(p_{\tau_D-} - \varepsilon))$. That is, starting at time τ_D, $\hat{D}$ reveals no information before either $p_{\tau_D-} + \varepsilon$ or $p_{\tau_D-} - \varepsilon$ is reached. Further, define $l_s^R = \frac{1-D_s^R}{1-D_{\tau_D-}^R}$ and $l_s^L = \frac{1-D_s^L}{1-D_{\tau_D-}^L}$. Then, Bayes' rule implies that

$$p_{\tau_D} = \frac{p_{\tau_D-} l_{\tau_D}^R}{p_{\tau_D-} l_{\tau_D}^R + (1 - p_{\tau_D-})\, l_{\tau_D}^L}.$$

Any optimal policy D must feature either $l_{\tau_D}^R = 1$ and $l_{\tau_D}^L < 1$ or $l_{\tau_D}^R < 1$ and $l_{\tau_D}^L = 1$, which follows from Lemma 18. Consider first the case $l_{\tau_D}^R < 1$ and $l_{\tau_D}^L = 1$. Define $l^R(\varepsilon)$ as

$$l^R(\varepsilon) = \frac{p_{\tau_D}}{1 - p_{\tau_D}} \frac{1 - p_{\tau_D-} - \varepsilon}{p_{\tau_D-} + \varepsilon}$$

and note that

$$p_{\tau_D} = \frac{(p_{\tau_D-} + \varepsilon)\, l^R(\varepsilon)}{(p_{\tau_D-} + \varepsilon)\, l^R(\varepsilon) + 1 - p_{\tau_D-} - \varepsilon},$$

i.e., given the prior $p_{\tau_D-} + \varepsilon$, using $l^R(\varepsilon)$ induces the posterior p_{τ_D} conditional on no disclosure. Given $l^R(\varepsilon)$, define $\hat{D}_{\tau(p_{\tau_D-}+\varepsilon)}^L = \hat{D}_{\tau(p_{\tau_D-}+\varepsilon)}^L = D_{\tau_D-}^L$, $\hat{D}_{\tau(p_{\tau_D-}+\varepsilon)}^R$ so that

$$l^R(\varepsilon) = \frac{1 - \hat{D}_{\tau(p_{\tau_D-}+\varepsilon)}^R}{1 - D_{\tau_D-}^R},$$

and $\hat{D}_{\tau(p_{\tau_D-}-\varepsilon)}^R$ so that

$$l^R(-\varepsilon) = \frac{1 - \hat{D}_{\tau(p_{\tau_D-}-\varepsilon)}^R}{1 - D_{\tau_D-}^R}.$$

The sender's continuation payoff under $\hat{D}$ is then given by

$$\hat{W}\left(p_{\tau_D-}\right) \equiv E\left[\int_{\tau_D}^{\tau(p_{\tau_D-}+\varepsilon)\wedge\tau(p_{\tau_D-}-\varepsilon)} e^{-(r+\lambda)(t-\tau_D)} w dt\right.$$

$$+e^{-(r+\lambda)\left(\tau(p_{\tau_D-}+\varepsilon)-\tau_D\right)} W\left(p_{\tau_D}\right) \mathbb{1}\{\tau(p_{\tau_D-}+\varepsilon)<\tau(p_{\tau_D-}-\varepsilon)\} \frac{\pi\left(\hat{D}_{\tau(p_{\tau_D-}+\varepsilon)}\right)}{\pi\left(D_{\tau_D-}\right)}$$

$$\left.+e^{-(r+\lambda)\left(\tau(p_{\tau_D-}-\varepsilon)-\tau_D\right)} W\left(p_{\tau_D}\right) \mathbb{1}\{\tau(p_{\tau_D-}-\varepsilon)<\tau(p_{\tau_D-}+\varepsilon)\} \frac{\pi\left(\hat{D}_{\tau(p_{\tau_D-}-\varepsilon)}\right)}{\pi\left(D_{\tau_D-}\right)}\right].$$

The same argument as in the proof of Proposition 9 (see Equation (59)) implies that

$$\frac{\pi\left(\hat{D}_{\tau(p_{\tau_D-}+\varepsilon)}\right)}{\pi\left(D_{\tau_D-}\right)} = p_{\tau_D-} l^R(\varepsilon) + 1 - p_{\tau_D-}$$

and

$$\frac{\pi\left(\hat{D}_{\tau(p_{\tau_D-}-\varepsilon)}\right)}{\pi\left(D_{\tau_D-}\right)} = p_{\tau_D-} l^R(-\varepsilon) + 1 - p_{\tau_D-}.$$

Since p_t follows Equation (7) for $t \in (\tau_D, \tau(p_{\tau_D-}+\varepsilon) \wedge \tau(p_{\tau_D-}-\varepsilon))$, it holds that

$$\Pr\left(\{\tau(p_{\tau_D-}+\varepsilon)<\tau(p_{\tau_D-}-\varepsilon)\right) = \Pr\left(\tau(p_{\tau_D-}-\varepsilon)<\tau(p_{\tau_D-}+\varepsilon)\right) = \frac{1}{2}.$$

Direct calculation shows that $l^R(\varepsilon)$ is convex in ε and hence

$$\frac{1}{2}\left(l^R(\varepsilon) + l^R(-\varepsilon)\right) \geq l^R(0) = l^R_{\tau_D}.$$

Thus,

$$\frac{1}{2}\left(\frac{\pi\left(\hat{D}_{\tau(p_{\tau_D-}+\varepsilon)}\right)}{\pi\left(D_{\tau_D-}\right)} + \frac{\pi\left(\hat{D}_{\tau(p_{\tau_D-}-\varepsilon)}\right)}{\pi\left(D_{\tau_D-}\right)}\right) \geq \frac{\pi\left(D_{\tau_D}\right)}{\pi\left(D_{\tau_D-}\right)}.$$

Together with the fact that $W(p_{\tau_D}) < \frac{w}{r+\lambda}$, this implies that $\hat{W}(p_{\tau_D-}) \geq W(p_{\tau_D-})$. That is, the strategy $\hat{D}$ is an improvement for the sender. The argument for the case $l^R_{\tau_D} = 1$ and $l^L_{\tau_D} < 1$ is analogous. $\square$

Lemma 20. *There exists a single interval $(\underline{p}, \bar{p})$ so that both players continue.*

Proof. There exists an interval $(\underline{p}, \bar{p}) \supset [\frac{c}{\lambda}, 1-\frac{c}{\lambda}]$ on which both players continue. Suppose that there exists another interval $(\underline{\hat{p}}, \bar{\hat{p}}) \subset [0, \frac{c}{\lambda})$ on which it is strictly optimal for both players to continue. Since $\lambda(1-p) > c$ for any $p \in (\underline{\hat{p}}, \bar{\hat{p}})$, stopping is dominated for player 2. Thus, it must be optimal for player 1 to stop at both $(\underline{\hat{p}}, \bar{\hat{p}})$. By Lemma 19, no disclosure is optimal on $(\underline{\hat{p}}, \bar{\hat{p}})$. This implies that player 1's value function satisfies the HJB equation (11) with boundary conditions $V_1(\underline{\hat{p}}) = V_1(\bar{\hat{p}}) = 0$, and that $V_1(p) > 0$ for $p \in (\underline{\hat{p}}, \bar{\hat{p}})$. Let $V_1(p)$ denote player 1's value when both player 1 and 2 continues forever. We have $\hat{V}_1(p) = \frac{\lambda p - c}{r+\lambda}$ and $V_1(p) \geq \hat{V}_1(p)$. Then Equation (11) implies that

$$\frac{1}{2}\sigma(p)^2 V_1''(p) = (r+\lambda) V_1(p) - (\lambda p - c) \geq 0$$

and hence $V_1''(p)$ is convex. But, then given $V_1(\underline{\hat{p}}) = 0$ and $V_1(p) > 0$ for $p \in (\underline{\hat{p}}, \hat{\bar{p}})$, it must be the case that $V_1'(p) > 0$ for all $p \in (\underline{\hat{p}}, \hat{\bar{p}})$, so that the boundary condition $V_1(\hat{\bar{p}}) = 0$ cannot hold. Thus, it cannot be optimal for player 1 to stop at $\hat{\bar{p}}$. An analog argument shows that no interval $(\underline{\hat{p}}, \hat{\bar{p}}) \subset (1 - \frac{c}{\lambda}, 1]$ exists on which it is optimal for both players to continue. □

Thus, players use the stopping strategies $\tau_1(\underline{p})$ and $\tau_2(\bar{p})$, and Proposition 19 implies that the sender's best response is given by the pipetting strategy $\{D^L(\underline{p}^*), D^R(\bar{p}^*)\}$. Given this strategy, player 1 and 2's value functions are characterized by Lemma 11 and 12. Given these value function, Lemma 13 implies that $\underline{p} = \underline{p}^*$ and $\bar{p} = \bar{p}^*$, since the fixed point in that Lemma is unique. Thus, the equilibrium characterized in Proposition 2 is unique.

A.3 Proof of Lemma 1

I first consider pipetting strategies.

Lemma 21. *For a given pipetting strategy $D^R(\hat{p})$, player 1's value function $V_1(p)$ is strictly decreasing in $\hat{p}$ for any p at which both players continue. Similarly, for a given pipetting strategy $D^L(\hat{p})$, player 2's value function $V_2(p)$ is increasing in $\hat{p}$ for any p at which both players continue.*

Proof. Fix $\bar{p} > 1 - \frac{c}{\lambda}$. Given $\bar{p}$, there exists a unique $\underline{p}_{ni}$, so that player 1 quits at $\underline{p}_{ni}$ if he sender provides no information (see Lemma 8 in the proof of Proposition 1).[37] Pick some $\hat{p} \in (\underline{p}_{ni}, \bar{p})$ and consider the pipetting strategy $D^R(\hat{p})$. Then, there exists a unique threshold $\underline{p} < \hat{p}$, so that player 1 quits whenever $p_t \leq \underline{p}$ (see Lemma 11 and 12). For $p \in (\underline{p}, \hat{p})$, $V_1(p)$ satisfies the HJB equation (11) with boundary conditions $V_1(\underline{p}) = V_1'(\underline{p}) = 0$ and $V_1'(\hat{p}) = \frac{1-V(\hat{p})}{1-\hat{p}}$. Pick some threshold $\underline{p}' > \underline{p}$ and consider a solution $\tilde{V}_1(p)$ to player 1's HJB equation (11) subject to the boundary conditions $\tilde{V}_1(\underline{p}') = \tilde{V}_1'(\underline{p}') = 0$. We have $V_1(\underline{p}') > \tilde{V}_1(\underline{p}') = 0$ and $V_1'(\underline{p}') > \tilde{V}_1'(\underline{p}') = 0$. Then, Equation (11) implies that $V_1''(\underline{p}') > \tilde{V}_1''(\underline{p}')$, which in turn implies that $V_1(p) > \tilde{V}_1(p)$, $V_1'(p) > \tilde{V}_1(p)$, and $V_1''(p) > \tilde{V}_1(p)$ for all $p > \underline{p}'$. This implies that

$$0 = V_1'(\hat{p}) - \frac{1 - V(\hat{p})}{1 - \hat{p}} > \tilde{V}_1'(\hat{p}) - \frac{1 - \tilde{V}_1(\hat{p})}{1 - \hat{p}}.$$

As in Lemma 12, we have $\tilde{V}_1''(p) > 0$, which implies that the function

$$\tilde{V}_1'(p) - \frac{1 - \tilde{V}_1(p)}{1 - p}$$

crosses zero at most once from below. This implies that if $\tilde{V}_1(p)$ satisfies the boundary condition

$$\tilde{V}_1'(\tilde{p}) = \frac{1 - \tilde{V}_1(\tilde{p})}{1 - \tilde{p}}, \tag{64}$$

[37] Note that this $\bar{p}_{ni}$ is not necessarily the same as in Proposition 1. This is immaterial for the proof.

it must do so at some $\tilde{p} > \hat{p}$. Conversely, the argument above implies that for any $\tilde{p} > \hat{p}$, the solution to player 1's HJB equation given the boundary condition (64) satisfies $V_2(\bar{p}') = V_2'(\bar{p}') = 0$ for some $\bar{p}' < \bar{p}$. Then, Lemmas 11 and 12 imply that this solution equals player 1's value function given the pipetting strategy $D^R(\hat{p})$. In particular, at $\hat{p}$ increases, $\underline{p}'$ increases and $V_1(p)$ decreases. The argument for player 2 is analogous. □

The Lemma above implies that when using pipetting strategies only, the maxmin strategy for player 1 sets $\hat{p} = \bar{p}$ and the maxmin strategy for player 2 sets $\hat{p} = \underline{p}$. It remains to show that no other strategy can induce a lower value for players 1 and 2.

Given $D^R(\bar{p})$, player 1 quits at threshold $\underline{p}$ and $V_1(p)$ is strictly increasing, strictly positive, and strictly convex for $p \in (\underline{p}, \bar{p})$. This immediately implies that any strategy with $dD_t^L > 0$ whenever $p_{t-} \in (\underline{p}, \bar{p})$ strictly increases player 1's value. Similarly, for $p_{t-} \in (\underline{p}, \bar{p})$, any strategy with $dD_t^R > 0$ strictly increases player 1's value. This follows immediately because of the boundary condition (64) (setting $\tilde{p} = \bar{p}$) and the fact that $V_1(p)$ is strictly convex and strictly increasing. Thus, the minmax strategy for player 1 is given by $D^R(\bar{p})$. The argument for player 2 is analogous.

Finally, it remains to show that $\{0, D^R(\bar{p})\}$ is a minmax strategy for player 1 and that $\{D^L(\underline{p}), 0\}$ is a minmax strategy for player 1. This follows immediately from the fact that $V_1(\underline{p}) = V_1'(\underline{p}) = 0$ under both $\{0, D^R(\bar{p})\}$ and $\{D^L(\underline{p}), D^R(\bar{p}\}\}$, which follows from Corollary 3. The result for player 2 follows from an analogous argument.

A.4 Proof of Proposition 3

Given two thresholds $(\underline{p}, \bar{p})$ such that $\underline{p} + \bar{p} = 1$, $W(p)$ is symmetric around $p = 1/2$. This follows immediately from the substitution $\hat{p} = 1 - p$ and the fact that $\sigma(p) = \sigma(1-p)$. Specifically, for $p_{0-} \in (\underline{p}, \bar{p})$, the sender's value function satisfies the ODE (13) with boundary conditions $W'(\bar{p}) + \frac{W(\bar{p})}{1-\bar{p}} = 0$ and $W'(\underline{p}) - \frac{W(\underline{p})}{\underline{p}} = 0$. Define $\hat{\bar{p}} = 1 - \underline{p}$ and $\hat{\underline{p}} = 1 - \bar{p}$ and note that $\hat{\bar{p}} + \hat{\underline{p}} = 1$. Then, substituting $\hat{p} = 1 - p$ into Equation (13) yields

$$(r + \lambda) W(\hat{p}) = w + \frac{1}{2}\sigma^2(\hat{p}) W''(\hat{p})$$

with the boundary conditions

$$W'\left(\hat{\bar{p}}\right) + \frac{W\left(\hat{\bar{p}}\right)}{1 - \hat{\bar{p}}} = 0$$

and

$$W'\left(\hat{\underline{p}}\right) - \frac{W\left(\hat{\underline{p}}\right)}{\hat{\underline{p}}} = 0.$$

Thus, the ODEs for p and $\hat{p}$ are identical with identical boundary conditions. This immediately implies that $W(p) = W(\hat{p}) = W(1-p)$. An analog argument applies to $W_{ni}(p)$, and establishes that $W_{ni}(p)$ is symmetric around $p = \frac{1}{2}$. In particular, $W(p)$ and $W_{ni}(p)$ both have their unique maximum at $p = 1/2$.

There exists a unique pair of points $\underline{p}_0^S$ and $\bar{p}_0^S$ so that $W_{ni}'(\underline{p}_0^S) = \frac{W_{ni}(\underline{p}_0^S)}{\underline{p}_0^S}$ and $W_{ni}'(\bar{p}_0^S) = -\frac{W_{ni}(\bar{p}_0^S)}{1-\bar{p}_0^S}$. This follows immediately from the fact that $W_{ni}(\underline{p}_{ni}) = W_{ni}(\bar{p}_{ni}) = 0$ and the fact that $W_{ni}(p)$ is strictly concave. Take $p = 0$ and the starting slope s_0, so that the line $s_0 p$ is tangent to

$W_{ni}(p)$ at $p = \underline{p}_0^S$. For any $s < s_0$, define with $\underline{p}_s$ the point so that the Robin boundary condition $W_s'(\underline{p}_s) = \frac{W_s(\underline{p}_s)}{\underline{p}_s}$ holds.

We have $\underline{p}_s > \underline{p}_0^S$. To see this, suppose by way of contradiction that $\underline{p}_s \leq \underline{p}_0^S$. Then, we have

$$s = W_s'\left(\underline{p}_s\right) = \frac{W_s\left(\underline{p}_s\right)}{\underline{p}_s} < s_0 = W_{ni}'\left(\underline{p}_0^S\right)\frac{W_{ni}\left(\underline{p}_0^S\right)}{\underline{p}_0^S}$$

and thus

$$W_s\left(\underline{p}_s\right) < W_{ni}\left(\underline{p}_0^S\right)\frac{\underline{p}_s}{\underline{p}_0^S} \leq W_{ni}\left(\underline{p}_0^S\right).$$

Moreover, since $s < s_0$, we have $W_s(p) < W_{s_0}(p) = W_{ni}(p)$ for all $p > \underline{p}_s$, and in particular $W_s''(p) < W_{s_0}''(p) = W_{ni}''(p)$. But this implies that $W_s'(p) < W_{s_0}'(p) = W_{ni}'(p)$, and in particular $W_s'(1/2) < W_{ni}'(1/2) = 0$. Thus, $W_s(p)$ cannot have a maximum at $p = 1/2$, a contradiction. Hence, it must be the case that whenever $s < s_0$, then $\underline{p} > \underline{p}_0^S$.

This result implies that if $\underline{p} > \underline{p}_0^S$, then $W(\underline{p}) < W_{ni}(\underline{p})$ and $W(p) < W_{ni}(p)$ for all $p \in (\underline{p}, \bar{p})$. An analog argument establishes that $W(p) < W_{ni}(p)$ for $p \in (\underline{p}, \bar{p})$ whenever $\bar{p} < \bar{p}_0^S$. Thus, $W(p) < W_{ni}(p)$ for all $p \in (\underline{p}, \bar{p})$ if $(\underline{p}, \bar{p}) \subset (\underline{p}_0^S, \bar{p}_0^S)$. Another analog argument establishes that $W(p) > W_{ni}(p)$ if $(\underline{p}_0^S, \bar{p}_0^S) \subset (\underline{p}, \bar{p})$.

A.5 Proof of Lemma 2

First, consider the pair $(\underline{p}_{1,mm}, \bar{p}_{1,mm})$ and the strategy $\{0, D^R(\bar{p}_{1,mm})\}$. For a fixed threshold $\bar{p}$, there exists a unique $\underline{p}$ so that player 1 stops at $\underline{p}$ given the strategy $\{0, D^R(\bar{p})\}$ (see the proof of Proposition 2). Moreover, $\underline{p}$ is increasing in $\bar{p}$ (see Equation (56)). Similarly, if player 1 stops at a fixed $\underline{p}$ and given no disclosure (i.e. $D^L = D^R = 0$), there exists a unique $\bar{p}$ so that player 2 stops at $\bar{p}$, and $\bar{p}$ is increasing in $\underline{p}$ (see the proof of Proposition 1). Thus, we can define a mapping $P(\bar{p})$, using player 1 and 2's problems just as in Lemma 13. A similar argument as in Lemma 13 implies that $P(\bar{p})$ is continuous and increasing, and that $P(1 - \frac{c}{\lambda}) > 1 - \frac{c}{\lambda}$ and $P(1) < 1$. Thus, a fixed point $P(\bar{p}) = \bar{p}$ exists. If multiple fixed points exist, choose the fixed point with the largest $\bar{p}$. In either case label the fixed point $\bar{p}_{1,mm}$, and label with $\underline{p}_{1,mm}$ the optimal stopping threshold for player 1 given $\bar{p}_{1,mm}$. Finally, note that in player 2's problem, I assumed that the sender provides no information. However, just as in the proof of Proposition 2, player 2's value is unchanged given pipetting at $\bar{p}_{1,mm}$, because of the boundary conditions $V_2(\bar{p}_{1,mm}) = V_2'(\bar{p}_{1,mm}) = 0$. Thus, $(\underline{p}_{1,mm}, \bar{p}_{1,mm})$ constitutes a fixed point given the strategy $\{0, D^R(\bar{p}_{1,mm})\}$ for both players. The proof for $(\underline{p}_{2,mm}, \bar{p}_{2,mm})$ given the strategy $\{D^L(\underline{p}_{2,mm}), 0\}$ is analogous.

A.6 Proof of Proposition 4

For any symmetric threshold strategy, we have $V_2(1-p) = V_1(p)$, which follows from using the substitution $\hat{p} = 1 - p$ as in the proof of Proposition 1. Thus, it is sufficient to consider player 1's incentives throughout. The following Lemma provides a comparison for the sender's value function

Lemma 22. *Consider two feasible threshold strategies with thresholds $(\underline{p}_C, \bar{p}_C)$ and $(\underline{p}_C', \bar{p}_C')$, with $(\underline{p}_C, \bar{p}_C) \subset (\underline{p}_C', \bar{p}_C')$. Let $W(p)$ be the sender's value given $(\underline{p}_C, \bar{p}_C)$ and $\hat{W}(p)$ be the sender's value* *given* $(\underline{p}'_C, \bar{p}'_C)$. *Then,* $\hat{W}(p) > W(p)$ *for all* $p \in (\underline{p}'_C, \bar{p}'_C)$.

Proof. The same argument as in Lemma 9 establishes that given $(\underline{p}_C, \bar{p}_C)$ the sender's value function $W(p)$ is the unique solution to the sender's HJB equation (13) with boundary conditions $W(\underline{p}_C) = W(\bar{p}_C) = 0$. Similarly $\hat{W}(p)$ is the unique solution to Equation (13) with boundary conditions $\hat{W}(\underline{p}'_C) = \hat{W}(\bar{p}'_C) = 0$. We have $\hat{W}(\underline{p}_C) > W(\underline{p}_C) = 0$. Suppose that there exists a $\tilde{p} \in (\underline{p}_C, \bar{p}_C)$ so that $W(p)$ hits $\hat{W}(p)$ from below, i.e. $W(\tilde{p}) = \hat{W}(\tilde{p})$ and $W'(\tilde{p}) > \hat{W}'(\tilde{p})$. Then, we have $W'(p) > \hat{W}'(p)$ for all $p > \tilde{p}$, since $W''(p) > \hat{W}''(p)$ whenever $W(p) > \hat{W}(p)$. But then, $\bar{p}_C > \bar{p}'_C$ which contradicts the assumption that $(\underline{p}_C, \bar{p}_C) \subset (\underline{p}'_C, \bar{p}'_C)$. Hence, it must be the case that $W(p) < \hat{W}(p)$. □

A threshold strategy is optimal if for some $p \in [\underline{p}_C, \bar{p}_C]$, both players are indifferent between continuing and stopping. Otherwise, the sender can increase the interval $(\underline{p}_C, \bar{p}_C)$, e.g. to $(\underline{p}_C - \varepsilon, \bar{p}_C + \varepsilon)$ and increase her payoff.

Lemma 23. *A symmetric threshold strategy is optimal only if* $\min_{p\in[\underline{p}_C,\bar{p}_C]} V_1(p) = \min_{p\in[\underline{p}_C,\bar{p}_C]} V_2(p) = 0$.

Proof. Denote with $\{D_t^L, D_t^R\}_{t\geq\tau(\underline{p}_C)}$ the sender's continuation policy once $\underline{p}_C$ is reached, where $\tau(\underline{p}_C) = \inf\{t \geq 0 : p_{t-} = \underline{p}_C\}$. Since the game ends with certainty once $\underline{p}_C$ or $\bar{p}_C$ are reached, the sender's value function satisfies the HJB equation (13) with boundary conditions $W(\underline{p}_C) = W(\bar{p}_C) = 0$. Denote similarly with $\{D_t^L, D_t^R\}_{t\geq\tau(\bar{p})}$ the continuation policy once $\bar{p}_C$ is reached, where $\tau(\bar{p}_C) = \inf\{t \geq 0 : p_{t-} = \bar{p}_C\}$. Suppose by way of contradiction that $\min_{p\in[\underline{p}_C,\bar{p}_C]} V_1(p) > 0$. Then, it is possible to pick thresholds $(\underline{p}_C - \varepsilon, \bar{p}_C + \varepsilon)$ so that given the same continuation policies at those thresholds, i.e. $\{D_t^L, D_t^R\}_{t\geq\tau(\underline{p}_C-\varepsilon)}$ and $\{D_t^L, D_t^R\}_{t\geq\tau(\bar{p}_C+\varepsilon)}$, we have $\min_{p\in[\underline{p}_C,\bar{p}_C]} V_1(p) \geq 0$. Then, since $(\underline{p}_C, \bar{p}_C) \subset (\underline{p}_C - \varepsilon, \bar{p}_C + \varepsilon)$, Lemma 22 implies that $\hat{W}(p) > W(p)$ for any $p \in (\underline{p}_C - \varepsilon, \bar{p}_C + \varepsilon)$. □

I now construct the optimal policy at the thresholds $\underline{p}_C$ and $\bar{p}_C$. Take $p_{t-} = \underline{p}_C$ and $p' \geq \bar{p}_{2,mm}$, where $\bar{p}_{2,mm}$ is the threshold at which player 2 stops given the minmax strategy $\{D^L(\underline{p}_{2,mm}), 0\}$ (see Lemma 2), and consider the policy D_t^L such that

$$p' = \frac{\underline{p}_C}{\underline{p}_C + (1 - \underline{p}_C)\frac{1-D_t^L}{1-D_{t-}^L}}.$$

This policy induces either beliefs $p_t = 0$ or $p_t = p'$. At $p_t = 0$, player 1 quits. At $p_t = p'$, the sender induces player 2 to quit by implementing the minmax strategy. Similarly, take $p_{t-} = \bar{p}_C$ and $p'' \leq \underline{p}_{1,mm}$, where $\underline{p}_{1,mm}$ is the threshold at which player 1 quits given the minmax strategy $\{0, D^R(\bar{p}_{1,mm})\}$ (see again Lemma 2), and consider the policy D_t^R such that

$$p'' = \frac{\bar{p}_C\frac{1-D_t^R}{1-D_{t-}^R}}{\bar{p}_C\frac{1-D_t^R}{1-D_{t-}^R} + 1 - \bar{p}_C}.$$

This policy induces either beliefs $p_t = p''$ or $p_t = 1$. At $p_t = 1$, player 2 quits. At $p_t = p''$, the sender induces player 1 to quit by implementing the minmax policy. Thus, under the given strategies, the game ends with probability one at the thresholds $\underline{p}_C$ and $\bar{p}_C$. Player 1's value at $\underline{p}_C$ and $\bar{p}_C$ is given by

$$V_1(\underline{p}_C) = \frac{\underline{p}_C}{p'} \in \left[\underline{p}_C, \frac{\underline{p}_C}{\bar{p}_{2,mm}}\right] \text{ and } V_1(\bar{p}_C) = \frac{\bar{p}_C - p''}{1-p''} \in \left[\frac{\bar{p}_C - \underline{p}_{1,mm}}{1-\underline{p}_{1,mm}}, \bar{p}_C\right],$$

while player 2's value is given by

$$V_2(\underline{p}_C) = \frac{p' - \underline{p}_C}{p'} \in \left[1 - \frac{\underline{p}_C}{\bar{p}_{2,mm}}, 1 - \underline{p}_C\right] \text{ and } V_2(\bar{p}_C) = \frac{1-\bar{p}_C}{1-p''} \in \left[1-\bar{p}_C, \frac{1-\bar{p}_C}{1-\underline{p}_{1,mm}}\right].$$

Symmetry requires that

$$V_1(\underline{p}_C) = \frac{\underline{p}_C}{p'} = \frac{1-\bar{p}_C}{1-p''} = V_2(\bar{p}_C)$$

and using $\underline{p}_C + \bar{p}_C = 1$ implies that $p'' + p' = 1$. This implies that

$$V_1(\bar{p}_C) = 1 - \frac{\underline{p}_C}{p'} = 1 - V_1(\underline{p}_C)$$

and similarly

$$V_2(\bar{p}_C) = 1 - V_2(\underline{p}_C).$$

In particular, defining $\hat{V} \in \left[\underline{p}_C, \frac{\underline{p}_C}{\bar{p}_{2,mm}}\right]$, it follows that

$$V_1(\underline{p}_C) = \hat{V} \text{ and } V_1(\bar{p}_C) = 1 - \hat{V}.$$

Lemma 24. *Any policy with $\hat{V} < \frac{\underline{p}_C}{\bar{p}_{2,mm}}$ is suboptimal.*

Proof. Since the policy is symmetric, it is sufficient to consider player 1 only. As in the proof of Proposition 1, define player 1's auxiliary problem as

$$\begin{aligned} V_1(p_{0-}) &= \sup_{\tau_1 \in \mathcal{T}} E\left[\int_0^{\tau_1 \wedge \tau(\bar{p}_C) \wedge \tau(\underline{p}_C)} e^{-(r+\lambda)t} u_1(p_t)\, dt + e^{-(r+\lambda)\tau(\bar{p}_C)}(1-\hat{V})\mathbb{1}\{\tau(\bar{p}_C) < \tau_1 \wedge \tau(\underline{p}_C)\}\right. \\ &\quad \left. + e^{-(r+\lambda)\tau(\underline{p}_C)}\hat{V}\mathbb{1}\{\tau(\underline{p}_C) < \tau_1 \wedge \tau(\bar{p}_C)\}\right]. \end{aligned}$$

Now, using the transformation $y = F(p)$ (see Equation (37)) and proceeding as in the proof of Proposition 1, define[38]

$$\bar{L}(y) = G_1(y) + \left(1-\hat{V}\right) y^{\frac{1}{2}\left(1+\frac{1}{A}\right)}\left(1 + y^{-\frac{1}{A}}\right) \text{ and } \underline{L}(y) = G_1(y) + \hat{V} y^{\frac{1}{2}\left(1+\frac{1}{A}\right)}\left(1 + y^{-\frac{1}{A}}\right).$$

Define $\bar{y} = F(\bar{p}_C)$ and $\underline{y} = F(\underline{p}_C)$. The function $\bar{L}(y)$ is the transformed payoff for $p \geq \bar{p}_C$ (or equivalently $y \geq \bar{y}$) when the game ends and player 1 receives $1-\hat{V}$ (see Equation (32)), and the function $\underline{L}(y)$ is the transformed payoff for $p \leq \underline{p}_C$ (or equivalently $y \leq \underline{y}$), when player 1 receives $\hat{V}$. The transformed stopping payoff is given by $G_1(y)$ in Equation (40). The functions $\underline{L}(y)$ and

[38] Notice that $\underline{L}(y) > \bar{L}(y)$ whenever $\hat{V} > 1/2$.

$\bar{L}(y)$ satisfy all properties listed in Lemma 3.

Consider the line connecting the points $(\underline{y}, \underline{L}(\underline{y}))$ and $(\bar{y}, \bar{L}(\bar{y}))$, i.e.

$$f(y) = \frac{\bar{y}\underline{L}(\underline{y}) - \underline{y}\bar{L}(\bar{y})}{\bar{y} - \underline{y}} + \frac{\bar{L}(\bar{y}) - \underline{L}(\underline{y})}{\bar{y} - \underline{y}} y.$$

If $f(y) > G_1(y)$ for all $y \in [\underline{y}, \bar{y}]$, then it is strictly suboptimal to stop for all $y \in [\underline{y}, \bar{y}]$. But this implies that $V_1(p) > 0$ for all $p \in [\underline{p}_C, \bar{p}_C]$ and the sender can improve her payoff as in Lemma 22.

Instead, assume instead that $f(y)$ is tangent to $G_1(y)$ at some point $y_1^* \in (\underline{y}, \bar{y})$, so that stopping at y_1^* is optimal for player 1. Then,

$$f(y) = \frac{\bar{y}G_1(y_1^*) - y_1^*\bar{L}(\bar{y})}{\bar{y} - y_1^*} + \frac{\bar{L}(\bar{y}) - G_1(y_1^*)}{\bar{y} - y_1^*} y$$

and the analog of Equation (3) holds, i.e.,

$$G_1'(y_1^*) = \frac{\bar{L}(\bar{y}) - G_1(y_1^*)}{\bar{y} - y_1^*}. \tag{65}$$

Now, consider the effect of increasing $\hat{V}$ on the line $f(y)$. We have

$$\frac{df(y)}{d\hat{V}} = -\frac{d\bar{L}(\bar{y})}{d\hat{V}} \frac{y_1^* - y}{\bar{y} - y_1^*} + \frac{dy_1^*}{d\hat{V}} \frac{df(y)}{dy_1^*}.$$

Using Equation (65) implies that $\frac{df(y)}{dy_1^*} = 0$. Thus, we have at $y = \underline{y}$

$$\frac{df(\underline{y})}{d\hat{V}} = \bar{y}^{\frac{1}{2}\left(1+\frac{1}{A}\right)} \left(1 + \bar{y}^{-\frac{1}{A}}\right) \frac{y_1^* - \underline{y}}{\bar{y} - y_1^*}.$$

At $y = \underline{y}$, we also have

$$\frac{d\underline{L}(\underline{y})}{d\hat{V}} = \underline{y}^{\frac{1}{2}\left(1+\frac{1}{A}\right)} \left(1 + \underline{y}^{-\frac{1}{A}}\right).$$

We have

$$\frac{d\underline{L}(\underline{y})}{d\hat{V}} > \frac{df(\underline{y})}{d\hat{V}}.$$

To see this, define $z(y) = y^{\frac{1}{2}\left(1+\frac{1}{A}\right)} \left(1 + y^{-\frac{1}{A}}\right)$ to save notation. The inequality is equivalent to

$$\frac{\bar{y}z(\underline{y}) + \underline{y}z(\bar{y})}{z(\underline{y}) + z(\bar{y})} > y_1^*.$$

Symmetry, in particular $\underline{p}_C = 1 - \bar{p}_C$, implies that $\underline{y} = \frac{1}{\bar{y}}$. Substituting this into the LHS above implies that

$$\frac{\bar{y}z(\underline{y}) + \underline{y}z(\bar{y})}{z(\underline{y}) + z(\bar{y})} = 1,$$

after some algebra. Lemma 3 implies that $y_1^* < \left(\frac{c}{\lambda-c}\right)^A < 1$, where the last inequality follows from $A > 1$ and $\lambda > 2c$.

Thus, increasing $\hat{V}$ implies that the line connecting the points $(\underline{y}, \underline{L}(\underline{y}))$ and $(\bar{y}, \bar{L}(\bar{y}))$ now lies strictly above $G_1(y)$ for all $y \in [\underline{y}, \bar{y}]$. But then, we have $V_1(p) > 0$ for all $p \in [\underline{p}_C, \bar{p}_C]$ and the sender once again has an improvement as in Lemma 22. Thus, any $\hat{V} \in (\underline{p}_C, \frac{\underline{p}_C}{\bar{p}_{2,mm}})$ is suboptimal. □

It remains to show that choosing D_t^L and D_t^R such that player 1's value at $\underline{p}$ is $\hat{V} = \frac{\underline{p}_C}{\bar{p}_{2,mm}}$ is optimal for the sender. That is, at $\underline{p}_C$, the sender either induces beliefs 0 or $\bar{p}_{2,mm}$, and at $\bar{p}_C$, the sender either induces beliefs $\underline{p}_{1,mm}$ and 1.

The same argument as in Lemma 18 implies that $D_t^L > D_{t-}^L$ if and only if $D_t^R = D_{t-}^R$. Consider the case when $D_t^R > D_{t-}^R$ and $p_{t-} = \underline{p}_C$. Then, since the game ends with probability one at $\underline{p}_C$ in a threshold equilibrium, it must be the case that

$$p' = \frac{\underline{p}_C \frac{1-D_t^R}{1-D_{t-}^R}}{\underline{p}_C \frac{1-D_t^R}{1-D_{t-}^R} + 1 - \underline{p}_C},$$

where $p' \leq \underline{p}_{1,mm}$. That is, the sender randomizes between $p_t = 1$ and $p_t = \underline{p}_{1,mm}$, and implements $\{0, D^R(\bar{p}_{1,mm})\}$ following $\underline{p}_{1,mm}$ so that player 1 quits. Player 1's value under this strategy is given by

$$V_1(\underline{p}_C) = \frac{\underline{p}_C - p'}{1 - p'} \in \left[\frac{\underline{p}_C - \underline{p}_{1,mm}}{1 - \underline{p}_{1,mm}}, \underline{p}_C\right].$$

Thus, under such a strategy, $V_1(\underline{p}_C) < \underline{p}_C < \frac{\underline{p}_C}{\bar{p}_{2,mm}}$. But Lemma 24 established that increasing player 1's payoff at $\underline{p}_C$ allows the sender to strictly increase her payoff. Thus, having $D_t^R > D_{t-}^R$ when $p_{t-} = \underline{p}_C$ is suboptimal. An analog argument shows that having $D_t^L > D_{t-}^L$ when $p_{t-} = \bar{p}_C$ is also suboptimal. Finally, if the equilibrium features no disclosure at $\underline{p}_C$, then $\hat{V} = 0$ and increasing $\hat{V}$ allows the sender to improve her payoff. Thus, the policy in the statement of Proposition 4 is the optimal symmetric threshold policy.

Given this policy, the sender's value function satisfies the HJB equation (13) with boundary conditions $W(\underline{p}_C) = W(\bar{p}_C) = 0$ and is strictly concave (see Lemma 9). Thus, there exist two points $\underline{\pi}$ and $\bar{\pi}$ such that $(\underline{\pi}, \bar{\pi}) \subset (\underline{p}_C, \bar{p}_C)$, so that the policy (D_0^L, D_0^R) satisfying

$$\underline{\pi} = \frac{\underline{p}_C}{\underline{p}_C + (1 - \underline{p}_C)(1 - D_0^L)} \text{ and } \bar{\pi} = \frac{\bar{p}_C(1 - D_0^R)}{\bar{p}_C(1 - D_0^R) + 1 - \bar{p}_C}$$

concavifies the sender's value function at $t = 0$.

A.7 Proof of Proposition 5

I first characterize the sender's best response and then show that given this best response, player 2 finds it optimal to continue at $\bar{p}$, whenever $\bar{p} < 1$.

Lemma 25. *If player 2 quits whenever $p_t \geq \bar{p}$, then the sender's best response is given by no disclosure for $p_{t-} \geq \bar{p}$ and D_t^L such that*

$$\bar{p} = \frac{p_{t-}}{p_{t-} + (1 - p_{t-}) \frac{1-D_t^L}{1-D_{t-}^L}} \tag{66}$$

whenever $p_{t-} < \underline{p}$.

Proof. Since the belief is Markovian it is sufficient to focus on p_{0-}. If $p_{0-} \geq \bar{p}$, Equation (30) implies that $W(p_{0-}) \leq 1$ for any policy $D \in \mathcal{D}$. Not disclosing any information (i.e. $D_t^L = D_t^R = 0$ for all $t \geq 0-$) attains the upper bound and is hence optimal.

Next, suppose that $p_{0-} < \bar{p}$. Taking $t = 0$ in Equation (66), the disclosure policy D_0^L induces belief $p_0 = \bar{p}$ with ex-ante probability $p_{0-} + (1 - p_{0-})(1 - D_0^L)$ and belief $p_0 = 0$ with the complementary probability. Given this policy, the sender's value is given by

$$\hat{W}(p_{0-}) = \frac{p_{0-}}{\bar{p}}.$$

Suppose by way of contradiction that there exists a nonempty interval $(\underline{p}, \bar{p})$ so that no disclosure is optimal whenever $p_{t-} \in (\underline{p}, \bar{p})$. Without loss of generality, suppose that there is no other interval to the left of $(\underline{p}, \bar{p})$ on which no disclosure is optimal. If $W(p) < \frac{p}{\bar{p}}$ for any $p \in (\underline{p}, \bar{p})$, then no disclosure is dominated by choosing D_t^L as in Equation (66) whenever $p_{t-} = p$. Thus, assume that $W(p) \geq \frac{p}{\bar{p}}$ for all $p \in (\underline{p}, \bar{p})$. On this interval, the sender's value satisfies the HJB equation

$$(r + \lambda) W(p) = \lambda p + \frac{1}{2}\sigma^2(p) W''(p), \tag{67}$$

which follows from a similar argument as in Lemma 1. We have

$$\frac{1}{2}\sigma^2(p) W''(p) = (r + \lambda) W(p) - \lambda p$$

so that $W''(p) > 0$ whenever $W(p) > \frac{\lambda p}{r+\lambda}$. We have $\frac{p}{\bar{p}} > \frac{\lambda p}{r+\lambda}$ and hence, $W(p) > \frac{\lambda p}{r+\lambda}$ so that $W''(p) > 0$ for all $p \in (\underline{p}, \bar{p})$. Since $W(\underline{p}) > \frac{\underline{p}}{\bar{p}} > 0$, and since there is no other interval with no disclosure to the left of $\underline{p}$, either the sender pippets at $\underline{p}$, i.e. uses the continuation strategy $D^L(\underline{p})$, or D_t^L exhibits a discrete jump when $p_{t-} = \underline{p}$. If the sender pippets at $\underline{p}$, then Proposition (**??**) implies that the sender's value function satisfies the boundary condition $W'(\underline{p}) = \frac{W(\underline{p})}{\underline{p}}$. But since $W(p)$ is strictly convex on $(\underline{p}, \bar{p})$, setting

$$D_t^L = \frac{\bar{p} - \underline{p}}{\bar{p}(1 - \underline{p})},$$

i.e. randomizing between beliefs $p_t = 0$ and $p_t = \bar{p}$ whenever $p_{t-} = \underline{p}$, induces an expected value of

$$\frac{\underline{p}}{\bar{p}} W(\bar{p}) > W(p),$$

for any $p \in (\underline{p}, \bar{p})$, so that no disclosure is suboptimal. Suppose instead that $D_t^L - D_{t-}^L > 0$ when $p_{t-} = \underline{p}$. This strategy either induces posteriors $p_t = 0$ or $p_t = \hat{p}$ for some $\hat{p} > \underline{p}$. The sender's value then equals

$$W(\underline{p}) = \frac{p}{\hat{p}} W(\hat{p}),$$

and since $W(p)$ is strictly convex, it must be the case that $W'(\underline{p}) < \frac{W(\hat{p})}{\hat{p}}$. But then, we have

$$\frac{p}{\hat{p}} W(\hat{p}) > W(p)$$

for all $p \in (\underline{p}, \bar{p})$. Thus, no disclosure for $p \in (\underline{p}, \bar{p})$ is again suboptimal.

Hence, for all $p < \bar{p}$, the policy D_t^L in Equation (66) dominates not disclosing information. Thus, any optimal policy by the sender induces a jump in beliefs for any $p_{0-} < \bar{p}$. The optimal such policy is given by Equation (66). □

Now, consider best response of player 2 at belief $\bar{p}$. If player 2 waits for a small amount of time, and the belief goes down to $\bar{p} - \varepsilon$, then the sender reveals information according to Equation (66). Thus, from player 2's perspective, the sender uses the left-pipetting strategy $D^L(\bar{p})$.

Lemma 26. *Given the left-pipetting strategy $D^L(\bar{p})$ for $\bar{p} < 1$, there exists a $\bar{p}' > \bar{p}$ so that it is optimal for player 2 to continue whenever $p \in (\bar{p}, \bar{p}')$.*

Proof. In the proof of Proposition 2, Corollary 2 implies that given the right-pipetting strategy $D^R(\bar{p})$, for any $\bar{p} > 0$, there exists a $y_1^* < \bar{y}$, where $\bar{y} = \left(\frac{\bar{p}}{1-\bar{p}}\right)^A$, so that stopping is optimal for player 1 whenever $y \leq y_1^*$ and continuing is optimal whenever $y \in (y_1^*, \bar{y}]$. In particular, as $\bar{y}$ goes to zero, then player 1's optimal stopping threshold (in y-space), y_1^* also goes to zero. As in the proof of Proposition 2, player 2's problem is equivalent to player 1's problem under the change of variable $\hat{p} = 1 - p$. Thus, given the left-pipetting strategy $D^L(\bar{p})$, there exists a $\bar{p}' > \bar{p}$ so that continuing is optimal for $p \in (\bar{p}, \bar{p}')$ and stopping is optimal for $p \geq \bar{p}'$. □

The Lemma implies that no MPE at which player 2 stops at some threshold $\bar{p} < 1$ can exist. Setting $\bar{p} = 1$ implies that the belief is degenerate conditional on $p_t = \bar{p}$. Then, player 2 quits and his value does not depend on the continuation strategy. Hence, $\bar{p} = 1$ is an MPE. It is also the unique MPE. Any MPE must feature an interval $[\bar{p}, 1]$ on which player 2 quits. Then, the above argument implies that the sender chooses D_t^L according to Equation (66), which implies that we must have $\bar{p} = 1$.

A.8 Proof of Proposition 6

For any threshold $\hat{p}$ so that player 2 quits at $\hat{p}$, the same argument as in the proof of Proposition 5 shows that setting

$$D_0^L = \frac{\hat{p} - p_{0-}}{\hat{p}(1 - p_{0-})}$$

is optimal at $t = 0$ whenever $p_{0-} < \hat{p}$. Then, the sender's optimal value is given by $W(p_{0-}) = \frac{p_{0-}}{\hat{p}}$ for all $p_{0-} < \hat{p}$ and $W(p_{0-}) = 1$ for $p_{0-} \geq \hat{p}$. By construction, $\bar{p}_{2,mm}$ is the lowest threshold at which player 2 quits for any continuation policy of the sender. Hence, the value $W(p_{0-}) = \frac{p_{0-}}{\bar{p}_{2,mm}}$ exceeds the sender's value given any other threshold $\hat{p}$ and given any other continuation policy. It is thus the sender's optimal value for any threshold policy.

B Additional Results

B.1 Case $\lambda < 2c$

Define $\bar{p} = c/\lambda$ and $\underline{p} = 1 - c/\lambda$, and note that $0 < \underline{p} < \bar{p} < 1$. Suppose that information arrives according to Equation (1) and that the sender does not disclose any information.

Conjecture the following equilibrium. For $p_t \in [\underline{p}, \bar{p}]$ both players quit with hazard rate $h_i(p_t)$ and are indifferent between continuing and quitting. For $p_t > \bar{p}$, player 2 quits immediately and player 1 continues. For $p_t < \underline{p}$, player 1 quits immediately and player 2 continues. Assume that $h_i(p)$ is Lipschitz continuous.

The value functions of each player satisfy the HJB equations

$$(r+\lambda) V_1(p) = \sup_{h_1 \geq 0} \lambda p + h_2(p) - c - (\lambda + h_2(p) + h_1) V_1(p) + V_1''(p) \frac{1}{2} \sigma(p)^2$$

and

$$(r+\lambda) V_2(p) = \sup_{h_2 \geq 0} \lambda(1-p) + h_1(p) - c - (\lambda + h_2 + h_1(p)) V_2(p) + V_2''(p) \frac{1}{2} \sigma(p)^2$$

on $[\underline{p}, \bar{p}]$.[39] Since players must be indifferent when they randomize, we have $V_1(p) = V_2(p) = 0$ for $p \in [\underline{p}, \bar{p}]$. This implies that

$$h_1(p) = c - \lambda(1-p)$$

and

$$h_2(p) = c - \lambda p.$$

In particular, we have the boundary conditions $V_i(\underline{p}) = V_i(\bar{p}) = 0$ for $i = 1, 2$.

For $p > \bar{p}$, the flow value of player 1 is $\lambda p + h_2 - c > 0$ for any $h_2 \geq 0$. Thus, continuing is a dominant strategy for player 1. But if player 1 continues with certainty, the flow value of player 2 is $\lambda(1-p) - c < 0$ for $p > \bar{p}$. Then, player 2 prefers to quit. For $p < \underline{p}$, the flow value for player 2 is $\lambda(1-p) + h_1 - c > 0$ for any $h_1 \geq 0$ and continuing is again a dominant strategy. Then, Player 1 quits. Thus, it must be the case that $V_1(p) = 1$ and $V_2(p) = 0$ for $p > \bar{p}$, and $V_1(p) = 0$ and $V_2(p) = 1$ for $p < \underline{p}$.

But this implies that player 1's value function is discontinuous at $\bar{p}$, and player 2's value function is discontinuous at $\underline{p}$. In other words, the value matching condition at $\underline{p}$ for player 1 and at $\bar{p}$ for player 2, i.e. $V_1\left(\underline{p}\right) = 1 = V_2(\bar{p})$ cannot hold. Thus, no such equilibrium exists.

[39] See Strulovici and Szydlowski (2015), Th. 1.