Investment Timing and Reputation

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Abstract

An agent learns dynamically about the profitability of a project and decides when to make an irreversible investment. The agent seeks to maximize his reputation for learning. Equilibrium strategies are dictated by the prior belief about the project's quality: a high-ability agent plays a dynamic cutoff strategy, where the cutoffs are bounded below by the prior. Agents are reputationally rewarded for both speed and accuracy, but accuracy becomes less consequential for reputation over time. Compared to a benchmark where the agent has no reputational motive, investment timing may be either premature or delayed. For projects with a large downside potential, reputation induces premature investment. Meanwhile, when projects have a positive net present value ex-ante, reputation induces delayed investment.

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1. Introduction

A firm's success can hinge on its ability to identify and invest in profitable new projects and technologies. This entails making investments that are at least in part irreversible. The question of when to optimally time an irreversible investment is a well-studied problem (Pindyck, 1991), but investment in R&D has two key features beyond irreversibility. First, firms will often not know whether a project will be successful when they invest. This means firms are not just deciding when to invest, but also whether to do so. Not only are R&D failures pervasive (Van der Panne, Van Beers, and Kleinknecht, 2003), for certain firms R&D investment may have a net negative effect on firm growth (Demirel and Mazzucato, 2012). Second, in practice such investment decisions are often made by CEOs and other managers who face career concerns. Indeed, the influence of CEO career concerns, and more specifically reputational concerns, on corporate investment decisions is well-documented empirically (Graham, Harvey, and Rajgopal (2005), Nadeem, Zaman, Suleman, and Atawnah (2021)).

In this paper, I study a reputation-driven agent who decides if and when to invest an a project of unknown quality. The agent learns dynamically whether the project is profitable and wishes to maximize his reputation for learning. My objective is two-fold. First, I aim to characterize the agent's investment behavior at a qualitative level. This will entail understanding the equilibrium relationship between an agent's investment behavior and reputation. Second, I ask how reputational motives can introduce inefficiencies in the timing of investment.

To this end, I present a model of irreversible investment under reputational concerns. In this model, an agent (e.g., CEO or manager) learns dynamically about a project's quality and decides if and when to make an irreversible investment in the project before some exogenous deadline. The agent may be either good, receiving an informative signal about project's quality in every period, or bad, receiving no information. In the baseline model, the agent's only objective is to maximize his reputation for learning, which is the belief held by the principal (e.g., the market) that he is of high ability. The agent's reputation is assessed by observing both his investment behavior and the project's quality.

I begin by characterizing the agent's equilibrium investment strategies. Under a weak selection assumption, strategies take a simple form: the good agent plays a cutoff strategy in every period, only investing if he is sufficiently confident that the project is profitable, while the bad agent mixes between investing and abstaining in every period. Due to the endogeneity of the agent's payoff function, namely his reputation, the prior belief about the project plays a crucial role in determining equilibrium strategies. In particular, the good agent's cutoff equals the prior belief in the last period, and strictly exceeds the prior in all previous periods. This implies that in some, if not all, periods, the agent is more willing to invest in projects that are ex-ante unprofitable. That is, reputational concerns induce the agent to invest in a way that is qualitatively inconsistent with profit maximization. I then characterize equilibrium reputation. I show that reputation in equilibrium has some intuitive properties: the agent is reputationally rewarded for making an accurate investment decision (i.e., investing when the project is profitable, and not investing when the project in unprofitable) and for speed conditional on investment being profitable. However, the effect of speed on reputation is subtle: conditional on making an unprofitable investment, the agent is penalized for speed. This conditional effect of speed on reputation implies that, while accuracy benefits the agent no matter when they invest, it becomes less consequential as time passes.

I then consider the distortionary effects this reputational motive can have on the timing of investment. To answer this question, I augment the agent's payoff function to be the weighted sum of two components: an intrinsic component, which captures profit, and a reputational component. I then compare the agent's investment behavior when there is no weight on the reputational component to that where there is a positive weight on reputation. I find that a reputational motive can cause investment to speed up or slow down, and which outcome arises depends on the nature of the investment problem. When there is a high downside potential from investment, in the sense that investing in an unprofitable project is very costly, reputational concerns cause the good agent to invest more quickly. This is due to the fact that under a high downside potential, the bad agent would never invest, even if the reputational reward from doing so is high. However, a good agent will still invest with positive probability because he may receive a strong enough signal that investment is profitable to make it worthwhile. Thus, investment must be reputationally rewarded, inducing the good agent's cutoffs to move leftward, and thus causing investment to speed up. However, whenever investment has a positive net present value under the prior, the opposite effect arises: reputational motives induce the agent to delay investment. In this case, the bad agent has a strict incentive to invest immediately in the absence of reputational concerns. So, when the reputational motive is small, the bad agent will still invest immediately, implying a reputational reward from abstaining, thus inducing the good agent to slow investment. Meanwhile, when the reputational motive is high, the bad agent invests with positive probability, implying that the reputation value of waiting is high enough that the good agent will want to slow investment as well. Because the good agent has a strictly higher reputational continuation value than the bad agent, this implies delayed investment. Together, these results suggest that reputational motives should induce hasty investment in precisely the sorts of projects where hasty investment is particularly costly and slow or under investment in projects where early investment is advantageous from an ex-ante perspective.

This paper contributes to the literature on real options models of investment, in which a decision maker makes an irreversible investment in the face of uncertainty. In canonical settings (Dixit and Pindyck (1994), McDonald and Siegel (1986)), this uncertainty pertains to the realizations of future flow payoffs, but not to the underlying data generating process. In contrast, I consider a decision maker who does not know this process, and thus whose option value comes in part from the ability to learn. In fact, there is a subset of this literature that incorporates dynamic learning (Bernanke (1983), Cukierman (1980), Décamps, Mariotti, and Villeneuve (2005)). In Bernanke (1983) and Cukierman (1980), a decision maker must choose one of several projects to pursue while learning about their relative values. Meanwhile, Décamps et al. (2005) considers a single project whose flow returns are dictated by a Brownian motion with unknown drift. As in my setting, they find that due to learning, the optimal stopping rule can entail investing after a drop in expected returns. I contribute to this literature primarily by studying an agent who is reputation-driven. In particular, I show that even under a pure reputational motive, the option value of investment arises endogenously as accuracy signals ability in equilibrium.

Thus, this paper contributes more precisely to the literature on investment timing with private learning under agency issues. In Bobtcheff and Levy (2017) and Bouvard (2014), an entrepreneur decides how long to experiment before investing in a project, where investment timing signals project quality and thus affects the chances of obtaining outside funding. Bouvard (2014) finds that investment is delayed under the equilibrium contract compared to first best, while Bobtcheff and Levy (2017) find that in a contract-free environment, agency issues cause hasty investment when learning is fast and delayed investment otherwise. More similar to this paper, Thomas (2019) and Grenadier and Malenko (2011) model an agent who derives utility from outsiders' beliefs about project quality. In Thomas (2019), the agent signals quality via her decision to abandon it, which leads to over-experimentation. Grenadier and Malenko (2011) provides a general model of signalling in a real options setting. They consider an application to investment in venture capital, finding that concerns about public perceptions of project quality yield hurried investment. In contrast to these papers, I model agent who plays a managerial role, tasked not with originating projects but rather with appraising them. Thus, the agent is not interested in signalling project quality, but instead in signalling ability to discern project quality. It is because of this reputation for learning that the direction of timing distortions in my setting depends on the nature of the investment problem, namely returns and ex-ante beliefs.

Finally, this paper connects to the literature on reputation for learning. Ottaviani and Sørensen (2006) present a general model of reputational cheap talk in a static setting. Meanwhile, Prendergast and Stole (1996) and Dasgupta and Prat (2008) present dynamic models of investment and trading where agents are motivated by both profit and reputation for learning. I contribute to this literature in two ways. First, I consider a real-options setting, studying the effect of reputational concerns on the timing of investment. Second, to my knowledge, this this the first paper to model a reputation for learning without assuming myopia. Namely, I consider a forward-looking agent who maximizes their long-term, rather than short-term, reputation. This forward-looking nature of the agent is precisely why there is an option value of investment even when the agent is purely reputation driven, and is also responsible for the equilibrium dynamics.

The rest of the rest of the paper proceeds as follows. In section 2, I present the baseline model where the agent's payoff is purely reputational. In sections 3 and 4, I characterize equilibrium investment strategies and reputation, respectively. In section 5, I augment the baseline model so that the agent places some weight on profit maximization, and analyze the effects of reputational concerns on investment timing. Finally, section 6 concludes. All formal proofs are relegated to the appendix.

2. Model

Fundamentals There is one agent and one principal. Time $t \in \{1, ..., T\}$ is discrete, with a finite horizon $T < \infty$. The state $\theta \in \{0, 1\}$ denotes whether investment in a project is profitable ($\theta = 1$) or unprofitable ($\theta = 0$). The agent and principal are endowed with a common interior prior $p_0 = Pr(\theta = 1) \in (0, 1)$. The agent is of type $i \in \{G, B\}$ (*good* or *bad*), which is time-invariant and independent of θ. The agent knows his type, but the principal holds a prior $R_0 \equiv Pr(i = G) \in (0, 1)$.

Learning The agent's type denotes his ability to learn about θ. Specifically, at the beginning of each t, an agent of type G observes some signal $s_t \in (0, \infty)$, distributed according to conditional density $f(\cdot|\theta)$. The signals s_t are labeled as their likelihood ratios, i.e., $s_t = \frac{f(s_t|\theta=1)}{f(s_t|\theta=0)}$. The s_t are i.i.d. across t given θ. I further assume $f(\cdot|\theta)$ is full support on $(0, \infty)$. Meanwhile, an agent of type B has no ability to learn: he observes no signal in any period.

Acting The agent, regardless of his type, chooses if and when to act (invest). Specifically, at each t (after observing s_t if $i = G$), the agent chooses $a_t \in \{\emptyset, 1\}$. $a_t = 1$ denotes *act* while $a_t = \emptyset$ denotes *abstain*. Acting is irreversible: if $a_t = 1$, then the agent is constrained to choose $a_s = 1$ for all $s > t$.[1] Thus, we can interpret acting as making an irreversible investment in the project. Let $\tau \in \{1, 2, ..., T, \emptyset\}$ denote the time at which the agent acts (i.e., the first t where $a_t = 1$), with $\tau = \emptyset$ denoting that the agent never acts.

Payoffs The agent's payoff, regardless of his type, is his reputation at the end of the game. This is the principal's belief that the agent is good, with knowledge of θ: $Pr(i = G|\tau, \theta)$. In assuming this belief is formed with knowledge of θ, I take the stance that the agent wishes to maximize his reputation in the long run, namely after the principal observes the state. This can be interpreted as assuming that the principal observes whether investment was profitable ex-post, i.e., after the agent makes his investment decision, and takes this into account when assessing the agent's ability.

Equlibrium A strategy for the good agent $A_t^G : [0, 1] \to [0, 1]$ specifies a probability of acting (choosing $a_t = 1$) at time t for every belief p, given that the agent has not yet acted (i.e., given $a_s = \emptyset$ for all $s < t$). [2]Meanwhile, a strategy for the bad agent $A_t^B \in [0, 1]$ denotes the probability of acting at t under belief p_0. A reputation function $R : \{1, ..., T, \emptyset\} \times \{0, 1\} \to [0, 1]$ denotes the principal's belief that $i = G$ given that the agent reported at τ and the state is θ. For any time-t signal history for the good agent, $(s_1, ..., s_t)$, let $P(s_1, ..., s_t)$ denote the agent's posterior after observing this signal history.

I seek a Markov perfect equilibrium of this game. This consists of strategies $\{A_i^t\}_{t=1}^T$ for each type, paired with a reputation function R and belief function P such that A_i maximizes $E_\theta[R(\tau, \theta)]$ at all (t, p) and both P and R are consistent with Bayes rule given (A_B, A_G).

Selection Because the agent's payoff depends only on his reputation and not intrinsically on the state, there exist a multiplicity of equilibria one may deem unintuitive. This includes both babbling equilibria and equilibria in which the good agent only acts when they are sufficiently certain that $\theta = 0$ (i.e., sufficiently sure that investment is unprofitable). To rule out such equilibria, I impose selection criterion (SC). To state this criterion, I must first define the agent's value function. Let $V_t^i(p, a)$ denote the type-i agent's time-t value from

[1] Equivalently, one can assume that once the agent chooses act, the game ends.

[2] In general, strategies could depend on the entire sequence of signals the good agent receives. However, it is without loss to restrict attention to Markov strategies within the class of equilibria that satisfy the selection criterion specified below.

playing $a_t = a \in \{\emptyset, 1\}$, given the agent has not yet acted (i.e., $a_s = \emptyset$ for all $s < t$). I now define (SC).

Definition 1. An equilibrium satisifes (SC) if

$$V_t^G(1,1) > V_t^G(\emptyset,1) \text{ and } V_t^G(\emptyset,0) > V_t^G(1,0). \text{ for all } t \in \{1,...,T\}.$$

(SC) imposes that an agent who knows $\theta = 1$ strictly prefers acting, while an agent who knows $\theta = 0$ strictly prefers abstaining. This implies that at the two extreme beliefs that the agent may hold, they act in line with standard notions of profit maximization (i.e., investing in profitable projects and not investing in unprofitable ones).[3] Note that given the above assumptions regarding the good agent's signal, these two beliefs obtain with probability zero. As I will show in what follows, this assumption is nonetheless sufficient to rule out babbling equilibria and ensure the equilibrium strategies take a simple form.

3. Equilibrium characterization

I now characterize the equilibrium. I begin by showing that any equilibrium that satisfies (SC) takes a qualitatively simple form. Then, as a stepping stone to a full characterization, I present the static characterization ($T = 1$). Finally, I characterize equilibrium strategies under the dynamic model ($T > 1$).

3.1. Equilibrium structure

Here, I show that in any equilibrium that satisfies (SC), the good agent plays a cutoff strategy while the bad agent mixes between acting and abstaining in every period. To establish this result, I rely on the convexity of the agent's continuation value in the belief. This property is stated as Lemma 1.

Lemma 1. *$V_t^G(p,\emptyset)$ is convex in p for all $t \in \{1,...,T\}$.*

Lemma 1 implies that, all else equal, an agent who receives a more conclusive signal has a greater continuation value in equilibrium. This result is intuitive: a more conclusive signal at time t ensures that the agent can more optimally choose whether to act or abstain in future periods for any path of future signal realizations, and thus yields a higher continuation value. Formally, this lemma follows from Blackwell's theorem and relies on the assumption that, conditional on the state, the agent's signals are independent over time.

[3] While I have not yet formalized profit, I will do this in section 5.

This ensures that a less Blackwell informative signal cannot yield a higher continuation value purely because the less informative signal is correlated with more informative ones in later periods.

I now qualitatively characterize the good and bad agent's strategies. This is stated as Proposition 1.

Proposition 1. *In any equilibrium that satisfies (SC), at all $t \in \{1, ..., T\}$:*

1. *There exists $p_t^* \in (0,1)$ such that* $A_t^G(p) = \begin{cases} 0 & \text{for all } p < p_t^* \\ 1 & \text{for all } p > p_t^* \end{cases}$
2. $A_t^B \in (0,1)$.

Proposition 1 states that in every period, there exists an interior cutoff belief such that the good agent acts (abstains) if his belief lies above (below) this cutoff. This results from Lemma 1 and (SC), and can be illustrated by a geometric argument. Figure 1 plots, for any given t, $V_t^G(p,1)$ and $V_t^G(p,\emptyset)$, i.e., the good agent's value from acting and abstaining, respectively. Now, let us make two observations. First, $V_t^G(p,1) = pR(t,1) + (1-p)R(t,0)$, is linear in the belief p, while $V_t^G(p,\emptyset)$ is convex in p (Lemma 1). Second, (SC) ensures that $V_t^G(p,1)$ lies strictly above $V_t^G(p,\emptyset)$ when $p=1$ and strictly below $V_t^G(p,\emptyset)$ when $p=0$. Together, these two facts imply that $V_t^G(p,1)$ intersects $V_t^G(p,\emptyset)$ at a unique interior point p_t^*, and thus that $V_t^G(p,1) > (<)\ V_t^G(p,\emptyset)$ to the right (left) of this point. So, a good agent who is acting optimally must employ a cutoff strategy of the form specified in Proposition 1.

Proposition 1 also asserts that the bad agent mixes between acting and abstaining in every period. To see why this is the case, suppose by contradiction the bad agent did not mix. Let t denote the first period where the bad agent plays a pure strategy. First, suppose the bad agent always abstains in period t ($A_t^B = 0$). Because s_t is full support over the likelihood ratios $\frac{f(s_t|\theta=1)}{f(s_t|\theta=0)}$, there is a strictly positive probability that the good agent's time-t belief p_t lies above p_t^*, and thus that the good agent acts. So, acting in period t reveals that the agent is good. The equilibrium reputation function must be consistent with this in equilibrium and assign a perfect reputation to an agent that acts in t: $R(t,\theta) = 1$ for $\theta \in \{0,1\}$. Furthermore, this perfect reputation holds regardless of the realization of the state. This is due to the fact that the good agent acts even when his belief p is interior, and thus acts with positive probability even when $\theta = 0$. Thus, $V_t^B(p_0,1) = 1$. Meanwhile, it must be that $V_t^B(p_0,\emptyset) < 1$: if this were not the case, the bad agent would always be earning a perfect reputation, implying that the reputation function R is inconsistent with the agent's strategies. Since $V_t^B(p_0,\emptyset) < V_t^B(p_0,1)$, the bad agent can profitably deviate by acting, instead of abstaining, in period t. Similarly, always acting ($A_t^B = 1$) implies that

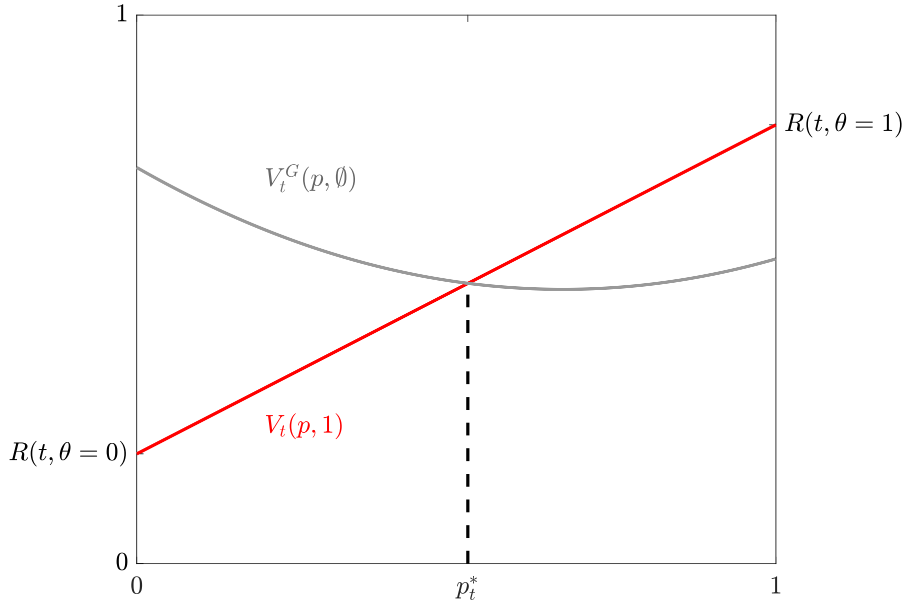

Figure 1: The good agent's value of acting ($V_t(p,1)$) and abstaining ($V_t(p,\emptyset)$), as a function of his belief p.

abstaining yields a perfect reputation, and thus abstaining becomes a profitable deviation. We conclude that the bad agent must mix between acting and abstaining in every period.

3.2. Static characterization

With Proposition 1 in hand, I now present the equilibrium characterization for the special case where $T = 1$. I first show that in the static equilibrium, the good agent acts if and only if his posterior about the state exceeds his prior. I state this as Claim 1. Throughout this section, I drop the time index from all functions and variables.

Claim 1. *When $T = 1$, there exists a unique equilibrium, under this equilibrium $p^* = p_0$.*

This results from the fact that, in a static setting, the good and bad agent enjoy the same value from both acting and abstaining at any given belief: $V_t^G(p,a) = V_t^B(p,a)$ for all beliefs p and $a \in \{\emptyset, 1\}$. Because there is a unique belief at which the good agent is indifferent between acting and abstaining, and the bad agent must be indifferent at p_0, it must be that this is the point of indifference for the good agent as well.

Before proceeding, let us take stock of this result. In a static setting, the good agent acts if and only if his posterior exceeds the prior. That is, an agent requires less confidence in the profitability of acting to do so when acting is unlikely to be profitable ex-ante. This holds despite the fact that the agent's prior p_0 provides no payoff-relevant information beyond that which is captured by his posterior p. Rather, the prior impacts the agent's equilibrium behavior via the reputation function: lowering the prior belief causes the equilibrium reputation function to adjust in such a way that the expected value of acting becomes relatively more profitable than that of abstaining for beliefs close to the prior, thus causing the agent's equilibrium cutoff to move leftward. More broadly, this result illustrates that a reputation-concerned agent's behavior has qualitative differences from that of an agent whose payoff function is exogenous. While in the latter case, the agent's prior will have no impact on his behavior beyond what is captured by the posterior, the agent's behavior is dictated by the prior when reputational concerns are present.

It remains to characterize the strategy of the bad agent. I show that the bad agent's strategy is also sensitive to the prior. Namely, A^B, is strictly increasing in the prior about the state. In other words, all else equal, the bad agent is less likely to invest when it is ex-ante unlikely that the project is profitable. This comparative static is formalized as Claim 2.

Claim 2. *Suppose $T = 1$, and fix an R_0 and $f(\cdot|\theta)$ for $\theta \in \{0, 1\}$. The bad agent's equilibrium probability of acting, A^B, is strictly increasing in p_0.*

This result follows from the fact that the bad agent mixes, and is thus indifferent between acting and abstaining, at his prior belief. Given any prior p_0, this implies

$$p_0[R(1, \theta = 1) - R(\emptyset, \theta = 1)] = (1 - p_0)[R(\emptyset, \theta = 0) - R(1, \theta = 0)].$$

It follows from the selection assumption that the agent enjoys a higher reputation from acting than abstaining when $\theta = 1$, and higher reputation from abstaining than acting when $\theta = 0$. Thus, holding fixed an equilibrium reputation function and increasing the prior makes acting relatively more valuable for the bad agent because $\theta = 1$ is more likely to realize. So to preserves indifference, when p_0 increases, the reputation function must adjust in such a way that acting is rewarded less. This will be achieved with a higher A^B: a higher A^B means the bad agent is relatively more likely to act, and thus that the equilibrium reputation from acting is lower regardless of which state is realized.

3.3. Dynamic characterization

Having characterized the static equilibrium, I now consider the dynamic case ($T > 1$). I begin by establishing existence of an equilibrium that satisfies selection, and consider qualitative features of the good agent's strategy. Namely, I show that in all periods before T, the good agent's cutoff lies strictly to the right of the prior. I formalize this result as Proposition 2

Proposition 2. *There exists an equilibrium that satisfies (SC). Under this equilibrium,* $p_T^* = p_0$ *and* $p_t^* > p_0$ *for all* $t < T$.

Equilibrium existence follows from the Kakutani fixed point theorem. Meanwhile, the upper bound on p_t^* follows from the fact that the good agent enjoys a higher continuation value than the bad agent in equilibrium, and a strictly higher continuation value when $t > T$. To fix ideas, let us start by considering the final period, T. As in the static setting, the good and bad agent have identical value functions, and thus, the two types of agent must be indifferent between acting and abstaining at the same belief. Because the bad agent mixes at his belief, this shared point of indifference must be the prior, p_0.

Now, let us consider an arbitrary period $t < T$. The two types of agent enjoy the same value from acting in period t at any given belief, $V_t(p, 1)$, but not identical continuation values. Specifically, $V_t^G(p, \emptyset) > V_t^B(p, \emptyset)$ for all $p \in (0, 1)$. This is because, unlike the bad agent, the good agent observes an informative signal about θ in $t+1$ (s_{t+1}). So, as long as the agent's optimal action depends on the state, i.e., $R(\tau, 0) \neq R(\tau, 1)$ for $\tau \in \{t+1, ..., T, \emptyset\}$, this signal will help the agent more optimally choose his stopping time and thus earn a strictly higher continuation value. Indeed, this is the case: (SC) asserts that the agent must enjoy a strictly higher value from acting (abstaining) when his belief is sufficiently close to 1 (0) and thus the optimal action does depend on the state. Because the good and bad agent enjoy the same value from acting but the good agent enjoys a strictly higher continuation value, the good agent requires a strictly higher belief to be indifferent between acting and abstaining. And thus, the good agent's point of indifference, p_t^*, must strictly exceed that of the bad agent, p_0.

This result can also be illustrated by a fairly simple geometric argument. Figure 2 plots the good agent's value as a function of his beliefs, as in Figure 1 for some $t < T$. Figure 2 also plots the value from acting in the next period $V_{t+1}(p, 1) = pR(t+1, 1) + (1-p)R(t+1, 0)$. Now, let us note two facts. First, this value lies strictly below the good agent's continuation value at time t, $V_t^G(p, \emptyset)$. This is due to the fact that, if the good agent continues in $t+1$, he can at least obtain the value from acting in $t+1$, and a strictly higher value by optimizing his strategy. Second, $V_t(p, 1)$ and $V_{t+1}(p, 1)$ must intersect at p_0: this is because the bad agent mixes in every period, which means that the agent must be indifferent between acting in periods t and $t+1$. These two facts together imply that $V_t^G(p, \emptyset)$ and $V_t(p, 1)$ must intersect strictly to the right of p_0, i.e., $p_t^* > p_0$.

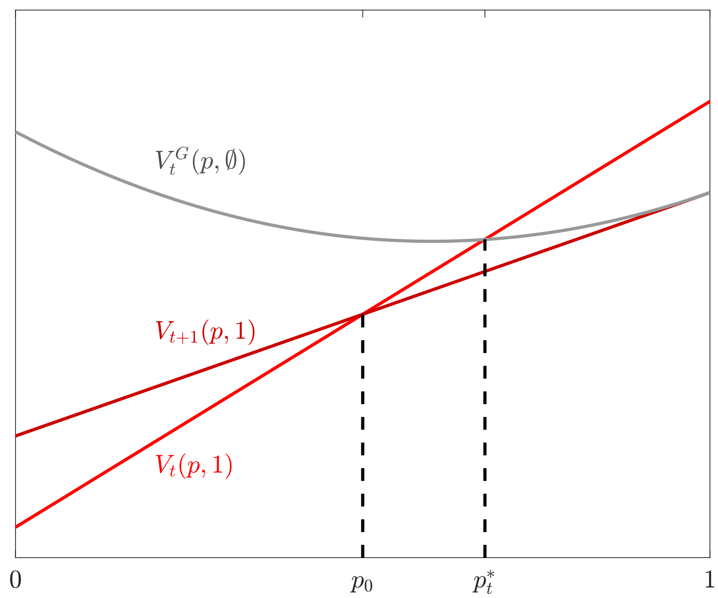

Figure 2: The good agent's point of indifference, p_t^*, lies strictly to right of the prior, p_0.

4. Reputation: speed and accuracy

In this section, I study the equilibrium reputation function. Specifically, I consider which qualities of a firm's report are reputationally rewarded, and how these may change over time. I find that the reputation function endogenously rewards accuracy in the agent's decision. The agent will also be rewarded for speed, but only conditional on making a correct decision. Namely, the agent suffers a greater reputational loss by making a mistake at earlier periods than later periods. I then argue this implies that accuracy becomes less important for reputation as time passes.

Let us now state the first result, namely that the agent is reputationally rewarded for accuracy. This is formalized as Proposition 3.

Proposition 3. *In any equilibrium that satisfies (SC):*

- $R(t, \theta = 1) > R(t, \theta = 0)$ *for all* $t \in \{1, ..., T\}$ *and*
- $R(\emptyset, \theta = 0) > R(\emptyset, \theta = 1)$.

Proposition 3 states that, no matter when the agent acts, he is reputationally better off if $\theta = 1$ (i.e., investment is profitable) than if $\theta = 0$ (i.e., investment is not profitable). Likewise, conditional on never acting, the agent is better off reputationally if $\theta = 0$ than if $\theta = 1$. That is, making an accurate decision, in the sense that the decision is the more profitable one, is beneficial for the agent's reputation.

This is a direct result of the qualitative nature of the good and bad agent's strategy (Proposition 1). Namely, because the good agent plays a cutoff strategy, his decision to act in any given period is correlated with the state: he is more likely to act if $\theta = 1$ than if $\theta = 0$. However, this is not the case for the bad agent: because he receives no signal in any period, his decision is necessarily uncorrelated with the state conditional on the prior belief. The equilibrium reputation function must account for this difference in correlation by assigning a higher reputation to an agent who makes the profit-maximizing decision.

This illustrates that even if an agent does not intrinsically benefit from making an accurate decision, he may nonetheless find it profitable to do so in equlibrium in order to signal his ability to learn. In fact, in a static setting, accuracy is the only tool an agent has to demonstrate his ability. But, in a dynamic setting, the agent can also use the timing of his action to signal his ability. In this regard, I show that in equlibrium, the reputation function strictly rewards speed if acting is the accurate decision (in the sense that $\theta = 1$), but strictly penalizes speed if acting is not the accurate decision (i.e., if $\theta = 0$). I formalize this as Proposition 4.

Proposition 4. *In any equilibrium that satisfies (SC):*

- $R(t, \theta = 1)$ *is strictly decreasing in* t*,*
- $R(t, \theta = 0)$ *is strictly increasing in* t

for $t \in \{1, ..., T\}$*.*

Let us first consider why $R(t, \theta = 1)$ is strictly decreasing in t. Recall from (SC) that the agent strictly prefers acting to abstaining at any t when $p = 1$. I.e., $V_t(1, 1) > V_t(1, 0)$. Further, because the good agent plays a cutoff strategy in every period, his value from continuing under belief $p = 1$ is equal to the value from acting in t: $V_t(1, \emptyset) = V_{t+1}(1, 1)$. Thus, $V_t(1, 1) > V_{t+1}(1, 1)$. Furthermore, because the value of acting at any t under belief $p = 1$ is just the reputation from acting under $\theta = 1$, it follows that $R(t, 1) > R(t + 1, 1)$.

That $R(t, 0)$ is strictly increasing in t follows from B's indifference condition. Namely, a bad agent who has not acted before t must be indifferent between acting in t and waiting until $t + 1$ to do so:

$$V_t(p_0, 1) = V_{t+1} \Leftrightarrow p_0 R(t, 1) + (1 - p_0) R(t, 0) = p_0 R(t + 1, 1) + (1 - p_0) R(t + 1, 0).$$

Since p_0 is interior, and acting at t yields a strictly higher reputation conditional on $\theta = 1$, this indifference can only be satisfied if acting at t yields a strictly lower reputation conditional on $\theta = 0$. I.e., $R(t, 0) < R(t + 1, 0)$. More concisely: the fact that acting earlier benefits the agent's reputation conditional on $\theta = 1$ means that in order for the bad type to be indifferent between acting and abstaining, acting earlier must harm the agent's reputation conditional on $\theta = 0$. Otherwise, acting earlier would yield a higher reputation regardless of the state, and thus the agent could profitably deviate by acting earlier, thus violating his indifference condition.

Proposition 4 tells us that speed's effect on reputation is subtle: while speed is reputation-improving for accurate decisions, it is reputation-damaging for inaccurate ones. This has implications for the importance of accuracy: Corollary 1 (below) states that the effect of the true state on reputation is higher the earlier the agent acts. In other words, while accuracy is beneficial to the agent's reputation no matter when the agent acts (in the sense that $R(t,1) - R(t,0)$ is positive for all t), its importance shrinks over time.

Corollary 1. *In equilibrium, $R(t,1) - R(t,0)$ is strictly decreasing in t for all $t \in \{1, ..., T\}$.*

While Proposition 4 establishes that speed has a positive effect on reputation when $\theta = 1$ and a negative effect when $\theta = 0$, it does not speak to the magnitudes of these effects. In fact, the relative magnitudes of these effects are dictated by the prior p_0: the higher the prior, the greater the positive effect of speed when $\theta = 1$ compared to the negative effect of speed when $\theta = 0$. This result is formalized as Corollary 2.

Corollary 2. *In equilibrium, for all $t \in \{1, ..., T-1\}$,*

$$\frac{R(t,1) - R(t+1,1)}{R(t+1,0) - R(t,0)} = \frac{p_0}{1-p_0}.$$

There is a simple explanation behind this result: if the prior increasese, $\theta = 1$ is more likely to realize ex-ante. So to preserve the bad agent's indifference between acting in t and $t+1$, the benefit of speed when $\theta = 1$, $R(t,1) = R(t+1,1)$, must decrease compared to the cost of speed when $\theta = 0$, $R(t+1,0) - R(t,0)$, to compensate for the fact that speed is more likely to be beneficial. Economically, this result illustrates that if investment is likely to be profitable ex-ante, then speed can do little to demonstrate that the agent is good, but can do significant reputational harm in the event of an error (i.e., investment in an unprofitable project). However, if investment is likely unprofitable ex-ante, then speed can be instrumental in positively showcasing the agent's ability, but can do little harm in the event of an error.

5. Timing distortions

In this section, I consider the distortionary effects reputational concerns can have on investment timing. To this end, I introduce a dual-objective payoff function where the agent receives an intrinsic payoff from profit in addition to a reputational payoff. I show that compared to a benchmark where the agent receives no reputational payoff, reputational concerns can cause investment to either speed up or slow down. In particular, whenever the downside potential is sufficiently high, in the sense that loss from investing in an unprofitable project is large, reputational concerns induce inefficiently early investment. However, when investment is ex-ante profitable, reputational concerns have the opposite effect: they induce delays in investment.

5.1. Dual-objective payoff

Let us begin by introducing the agent's augmented payoff function, $\tilde{U}$:

$$\tilde{U}(\tau, \theta) = (1 - X)\beta^{\tau} K_{\theta} \mathbb{I}(\tau \neq \emptyset) + XR(\tau, \theta),$$

where the K_{θ}, X, and β are parameters such that

$$K_1 > 0, K_2 < 0, X \in [0, 1], \beta \in (0, 1).$$

Under $\tilde{U}$, the agent's payoff is a convex combination of two components. The first, $\beta^{\tau} K_{\theta} \mathbb{I}(\tau \neq \emptyset)$, is the profit from investment. Namely, investment is profitable if and only if $\theta = 1$ ($K_1 > 0 > K_0$). The payoff from never investing is normalized to zero. In addition, the profit from investing is geometrically discounted: delaying investment results in lower profits when the project is good, but also lower losses when the project is bad. The second component of the payoff function is the agent's reputational payoff, as specified in section 2. Thus, this payoff fucntion specifies a dual objective: the agent cares about maximizing profit, but also maximizing reputation, where $X \in [0, 1]$ specifies the weight the agent places on his reputational payoff.[4] The payoff function specified in section 2 is a special case of the payoff in which no weight is placed on profit maximization, where $X = 1$. Except for this modified payoff function, I maintain all the assumptions of section 2, including the selection assumptions.

[4] Similar dual-objective payoff functions appear in other papers that study the effect of reputational concerns on investment, including Prendergast and Stole (1996).

5.2. Benchmark: optimal rule without reputation

As a benchmark, I begin by characterizing the agent's optimal stopping rule when he exclusively cares about profit maximization (i.e., when $X = 0$). This is formalized as Proposition 5.

Proposition 5. *When $X = 0$, the optimal stopping rule is the following:*

- *The bad agent acts in any period if and only if $p_0 < \hat{p} \equiv \frac{-K_0}{K_1 - K_0}$ for all t:*

$$A_t^B = \begin{cases} 1 & \text{if } p > \hat{p} \\ 0 & \text{if } p < \hat{p}. \end{cases}$$

- *The good agent plays a cutoff rule in every period:*

$$A_t^G(p) = \begin{cases} 1 & \text{if } p > \hat{p}_t \\ 0 & \text{if } p < \hat{p}_t, \end{cases} \tag{1}$$

where the $\hat{p}_t \in (0, 1)$ are unique and strictly decreasing in t.

Proposition 5 states that in the absence of reputational concerns, the bad agent acts immediately if p_0 is sufficiently high, and otherwise never acts. This is due to the fact that the bad agent is unable to learn. Because there is discounting in the payoff from acting, if his prior is such that acting is optimal, he will do so immediately and otherwise will abstain indefinitely. Meanwhile, the good agent employs a cutoff rule in every period, where the cutoffs are strictly decreasing with time. I.e., the agent becomes more willing to act as time passes. The decreasing nature of the good agent's cutoffs is due to the non-stationarity of his problem: the closer the good agent gets to the deadline T, the less time he has left to learn, and thus the lower his continuation value is at any given belief. Hence, the agent will find it optimal to act for a wider range of beliefs as time passes.

5.3. Impact of reputation on stopping time

Now, I consider how reputational concerns can cause deviations from the optimal rule established above. Namely, I show that a positive weight on reputation in the dual-objective payoff function can cause both accelerated and delayed investment. I begin by showing that whenever investment in the bad project is sufficiently costly, reputational concerns will cause the good agent to speed up investment. I formalize this as Proposition 6. But first, I introduce some notation. Let $(\hat{p}_t)_{t=1}^T$ denote the optimal cutoffs in the no-reputation case ($X = 0$), and let $(p_t^*)_{t=1}^T$ denote the equilibrium strategy of the agent when $X > 0$.

Proposition 6. *Fixing all other parameters and assuming $X > 0$, there exists a $\underline{K}$ such that if $K_0 < \underline{K}$, $\hat{p}_t > p_t^*$ in any equilibrium.*

Proposition 6 states that for any X, if K_0 is sufficiently negative, the good agent's cutoff shifts leftward from the no-reputation benchmark in every period. This implies that the good agent is not only more likely to act, he is more likely to act in earlier periods. Formally, it is an immediate corollary of Proposition 6 that the distribution of stopping times without reputational concerns first-order stochastically dominates any distribution of stopping times with reputational concerns.

Let us now consider the reasoning behind this proposition. To this end, it is helpful to decompose the agent's equilibrium value function into two components:

$$V_t^i(p, a) = (1 - X)V_t^{NR,i}(p, a) + XV_t^{R,i}(p, a),$$

where

$$V_t^{NR,i}(p, a) \equiv E_{\tau,\theta}[\beta^\tau K_\theta \mathbb{I}(\tau \neq \emptyset)|(p_s^*)_{s=t+1}^T, a_t = a]$$
$$V_t^{R,i}(p, a) \equiv E_{\tau,\theta}[R(\tau, \theta)|(p_s^*)_{s=t+1}^T, a_t = a].$$

$V_t^{NR,i}$ denotes the non-reputational value (i.e., value from profit), while $V_t^{R,i}$ denotes the reputational value. Note that if K_0 is sufficiently negative, the bad agent never acts in equilibrium, even if there is a reputational benefit from doing so: the expected loss in profit from not acting at any t, $V_t^{NR,B}(p_0, 1) - V_t^{NR,B}(p_0, \emptyset)$ is so large that any potential reputational gain from acting, $V_t^{R,B}(p_0, 1) - V_t^{R,B}(p_0, \emptyset)$ cannot possibly compensate the bad agent enough to make acting optimal. Meanwhile, the good agent plays an interior cutoff strategy in every period regardless of K_0, and thus acts with positive probability in every period. Thus, any equilibrium reputation function yields a perfect reputation from acting, and less-than-perfect reputation from abstaining, regardless of the state. This implies the good agent's reputational value from acting exceeds that from abstaining for any belief:

$$V_t^{R,G}(p, 1) > V_t^{R,G}(p, \emptyset) \text{ for all } p \in [0, 1].$$

Hence, the good agent strictly prefers acting at the non-reputational optimal cutoff $\hat{p}_t$: while the non-reputational value yields the good agent indifferent at the this belief, the reputational value yields acting strictly optimal. So, the good agent's cutoff shifts leftward in light of this reputational payoff.

Let us now economically interpret this result. Proposition 6 suggests that for projects that have a sizeable downside potential – in the sense that there is a substaintial profit loss if it turns out to be a bad investment – reputational concerns induce premature investment. Intuitively, this downside potential ensures that investment serves as a highly costly – and thus credible – signal that the agent is good. Indeed, the signal is so costly that the bad agent will never invest: only the good agent can be confident enough that the project is good enough to make investment worthwhile. This induces good investors to act prematurely, as investment can bolster their reputations, even if it turns out that the project was unprofitable. That is, reputation not only makes investment worthwhile, it also hedges the profit risk of investment, providing the agent with a further incentive to invest when it is otherwise not optimal.

A relatively high downside potential was essential to establishing hurried investment in Proposition 6. Intuitively, this would suggest that if the opposite were true – namely that there is a relatively high upside potential – investment should be delayed as a result of reputation. Indeed, one can show that if the ex-ante expected profit from investment is positive – due to a low downside potential (K_0), high upside potential (K_1), or high prior (p_0) – reputation causes delays in investment. This is established as Proposition 7.

Proposition 7. *Suppose that $p_0 > \hat{p}$ and that $X > 0$. In any equilibrium, $p_t^* \geq \hat{p}_t$, where the inequality holds strictly when $t = 1$.*

Formally, Proposition 7 states that whenever the prior belief is such that investment is profit maximizing, reputational concerns cause the good agent's cutoffs to move rightward. This implies that the good agent slows down his action: it is an immediate corollary of Proposition 7 that the distribution of stopping times when $X > 0$ first-order stochastically dominates that without. Like in Proposition 6, the agent in this case signals their ability by acting in a way that is inconsistent with profit maximization, and thus costly profit-wise, for the bad agent. However, when investment is profitable ex-ante, this entails under-investment rather than over-investment.

To better understand this proposition, it is helpful to first consider the static case $(T = 1)$. If $p_0 > \hat{p}$, the bad agent has a strict incentive to act in the absence of reputational concerns. I.e., $V_t^{NR,B}(p_0, 1) > V_t^{NR,B}(p_0, \emptyset)$. This means that if reputational concerns are small, the bad agent will always act in equilibrium. So, the equilibrium reputation function must reward abstaining and thus the reputational value from abstaining must exceed that from acting at any belief:

$$V_t^{R,G}(p,\emptyset) < V_t^{R,G}(p,\emptyset) \text{ for all } p.$$

Thus, the good agent must strictly prefer acting at the non-reputational optimal cutoff $\hat{p}$, implying that the equilibrium cutoff p lies to the right of $\hat{p}$. Meanwhile, if reputational concerns are sufficiently large, the bad agent must mix between acting and abstaining in equilibrium, implying tha the good agent's equilibrium cutoff is p_0, again lying to the right of $\hat{p}$.

The above reasoning can be extended to a dynamic setting, except for two differences. First, if the bad agent is indifferent at some $t < T$, the good agent's point of indifference will not coincide with the prior, but rather lie to the right. Second, it is possible that reputational concerns are so small that the bad agent doesn't mix and rather acts with probability 1 at some $t < T$. In such cases, the good agent who continues past t will enjoy a perfect reputation regardless of what actions he takes thereafter. Thus, the good agent will employ the profit-maximizing cutoffs $\hat{p}_s$ for all $s > t$. It is for this reason that Proposition 7 includes the caveat that p_t^* may not strictly exceed $\hat{p}_t$ in periods beyond the first.

Let us now take stock of these results. Together, Proposition 6 and Proposition 7 establish that reputational concerns can induce both hurried and delayed investment. Bobtcheff and Levy (2017) similarly find that both types of distortions are possible in their environment. However, while they find that the type of distortion is dictated by the speed of learning, I find that under reputational concerns, it is rather the fundamentals of the investment problem that determine the nature of timing distortions. This is because distortions arise in such a way that induce a reputation-profit tradeoff for the bad agent: reputation-improving distortions in the timing of investment must entail an expected loss in profits in equilibrium. Namely, while signalling ability via investment timing is not intrinsically costly for profit, it is necessarily costly in equilibrium.

6. Conclusion

I study a reputation-driven agent who learns dynamically about the profitability of a project and decides when to make an irreversible investment. Unlike models without reputational concerns, the equilibrium strategy of the agent is determined by the prior belief about the profitability of investment: in at least some periods, the agent is more likely to invest in projects that are less likely to be profitable ex-ante. In equilibrium, the agent is reputationally rewarded for both accuracy and speed, but accuracy becomes less consequential for reputation with time. Furthermore, speed is beneficial only conditional on the agent making the correct investment decision and is otherwise harmful, with the relative size of this harm increasing in the prior belief. I then suppose the agent is motivated by both profit and reputation to understand the effect of reputation on investment timing. I find that reputation can cause both hurried and delayed investment, and that which sort of distortion arises is determined by the nature of the investment problem: hurried investment obtains for projects with a relatively high downside potential, whereas delayed investment arises for projects with a positive net present value ex-ante.

In the model I present, I make two stark assumptions. First, the agent exclusively cares about their reputation at the end of the game, and places no weight on their interim reputation prior to the deadline. Second, I assume that the true state is revealed regardless of whether the agent decides to invest in the project. Both these assumptions are consequential for the qualitative nature of the equilibria, and one can easily imagine economic environments in which they do not hold. Characterizing the equilibrium while relaxing these assumptions is the subject of ongoing work.

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7. Proofs

Before proceeding, let us define two different conditional distributions. First, let $G_t(\cdot|p_{t-1})$ denote the good agent's distribution of time-t beliefs given that their time $t-1$ belief was p_{t-1}. It follows from the definition of F that on-path in any equilibrium:

$$G_t(p_t|p_{t-1}) = F((\frac{1-p_{t-1}}{p_{t-1}})(\frac{p_t}{1-p_t})).$$

Second, let $H_t(\cdot|p_{t-1})$ denote the distribution of time-t beliefs given $\tau \notin \{1, ..., t\}$ and that their time-$t-1$ belief was p_{t-1}. Finally, let $H_t(\cdot)$ denote the good agent's distribution of time-t beliefs given $\tau \notin \{1, ..., t\}$, conditional on p_0 (namely, not conditional on the time-$t-1$ belief). It is computed recursively as follows:

$$H_1(p_1) = H_1(p_1|p_0)$$

$$H_t(p_t) = \int_0^1 H_t(p_t|p_{t-1})dH_{t-1}(p_{t-1}).$$

Proof of Lemma 1. Proof by induction.

Base case: $t = T$. Note that

$$V_T^G(p, \emptyset) = pR(\emptyset, 1) + (1-p)R(\emptyset, 0),$$

which is linear in p and thus convex in p.

Induction step: Fix any $t < T$, and assume by induction that $V_{t+1}(p, \emptyset)$ is convex in p. We want to show that $V_t(p, \emptyset)$ is convex in p. This is equivalent to showing that for all $p, p' \in [0, 1]$ and $\lambda \in [0, 1]$,

$$\lambda V_t(p, \emptyset) + (1-\lambda)V_t(p', \emptyset) \geq V_t(\overline{p}),$$

where $\overline{p} \equiv \lambda p + (1-\lambda)p'$. To this end, fix a p, p', and $\lambda \in [0, 1]$ and define the following binary signal $b \in \{0, 1\}$ on θ:

$$Pr(b = 1|\theta = 1) = \frac{p\lambda}{\overline{p}}, \qquad Pr(b = 1|\theta = 0) = \frac{(1-p)\lambda}{1-\overline{p}},$$

Now define the following two signals σ and $\tilde{\sigma}$:

$$\sigma : \{0, 1\} \to \Delta([0, \infty]), \text{ where } \sigma(\theta) = F(\cdot|\theta)$$

$$\tilde{\sigma} : \{0, 1\} \to \Delta([0, \infty] \times \{0, 1\}), \text{ where } \tilde{\sigma}(\theta) = \tilde{F}(\cdot, \cdot|\theta),$$

and for $b^* \in \{0, 1\}$, $\tilde{F}(s, b^*) = F(s|\theta)Pr(b \leq b^*|\theta)$. Note that $\tilde{\sigma}$ Blackwell dominates σ. Now assuming that the agent has prior belief p let $G_t(\cdot|p)$ and $\tilde{G}_t(\cdot|p)$ denote the distribution of posteriors after observing σ and $\tilde{\sigma}$, respectively. It follows from the Law of Iterated Expectations that:

$$\tilde{G}_t(q|\overline{p}) = Pr(b = 1|\overline{p})G_t(q|p) + Pr(b = 0|\overline{p})G_t(q|p') = \lambda G_t(q|p) + (1-\lambda)G_t(q|p'). \quad (2)$$

Now, note that since $V_{t+1}(p, \emptyset)$ is convex in p and $V_{t+1}(p, 1) = pR(t+1, 1) + (1-p)R(t+1, 0)$

is linear in p, $V_{t+1}(p) = \max\{V_{t+1}(p, \emptyset), V_{t+1}(p, 1)\}$ is also convex in p. Thus

$$\begin{aligned} V_t(\emptyset, \overline{p}) &= \int_0^1 V_{t+1}(q) dG_t(q|\overline{p}) \\ &\leq \int_0^1 V_{t+1}(q) d\tilde{G}_t(q|\overline{p}) \\ &= \lambda \int_0^1 V_{t+1}(q) dG_t(q|p) + (1-\lambda) \int_0^1 V_{t+1}(q) dG_t(q|p') \\ &= \lambda V_t(p, \emptyset) + (1-\lambda) V_t(p', \emptyset), \end{aligned}$$

where the inequality follows from Blackwell's (1953) theorem, and the second equality follows from (2). □

Before proceeding, let us define the agent's *interim reputation* as follows:

Definition 2 (Interim reputation)**.** The agent's time t interim reputation is the principal's belief $i = G$ given that they did not report at or before t:

$$R_t \equiv Pr(i = G | \tau \notin \{1, ..., t\}).$$

Lemma 2. *In any equilibrium, if for all $s \leq t$ there exists a $p_s^* \in (0, 1)$ such that*

$$A_t^G(p) = \begin{cases} 0 & \textit{for all } p < p_t^* \\ 1 & \textit{for all } p > p_t^* \end{cases}$$

and $A_t^B \in (0, 1)$, then $R_t \in (0, 1)$.

Proof. Fix a t, and assume A^G and A^B satisfy the assumptions specified in Lemma 2. We want to show that $R_t \in (0, 1)$. Proof by induction.

Base case: $s = 0$. $R_s = R_0 \in (0, 1)$ by assumption.

Induction step: For any $s \in \{1, ..., t\}$, assume $R_{s-1} \in (0, 1)$. We want to show that $R_s \in (0, 1)$. It follows from Bayes Rule that

$$R_s = \frac{1}{1 + \frac{Pr(\tau \neq s | \tau \notin \{1, ..., s_{t-1}\}, i = B)}{Pr(\tau \neq s | \tau \notin \{1, ..., s_{t-1}\}, i = G)}}. \quad (3)$$

To show that $R_s \in (0, 1)$, it suffices to show that both the conditional probabilities in (3) lie in $(0, 1)$. In equilibrium,

$$Pr(\tau \neq s | \tau \notin \{1, ..., s_{t-1}\}, i = B) = A_t^B \in (0, 1),$$

where $A_t^B \in (0,1)$ holds by assumption. It remains to show that $Pr(\tau \neq s|\tau \notin \{1,...,s_{t-1}\}, i = G) \in (0,1)$. To this end, because the good agent is playing a cutoff strategy,

$$H_t(p_t|p_{t-1}) = \begin{cases} 0 & \text{for all } p < p_t^* \\ \frac{G_t(p_t|p_{t-1}) - G_t(p_t * |p_{t-1})}{G_t(p_t^*|p_{t-1})} & \text{for all } p > p_t^* \end{cases}$$

We can write

$$Pr(\tau \neq s|\tau \notin \{1,...,s-1\}, i = G) = \int_0^1 G_t(p_t^*|p_{t-1}) dH_{t-1}(p_{t-1}). \tag{4}$$

Now, we make two observations:

1. $G_t(p_t^*|p_{t-1}) \in (0,1)$ for all $p_{t-1} \in (0,1)$.
2. $H_{t-1}(p_{t-1})$ is continuous in p_{t-1}, following from the continuity of $G_{t-1}(p_{t-1}|p_{t-2})$ in p_{t-1}.

It follows from the above two observations, combined with (4) that $Pr(\tau \neq s|\tau \notin \{1,...,s-1\}, i = G) \in (0,1)$.

□

Proof of Proposition 1. Fix any t. By (SC), $V_t(1,\emptyset) > V_t(1,1)$ and $V_t(0,\emptyset) > V_t(0,1)$. Because $V_t(p,\emptyset)$ is convex in p (Lemma 1) and $V_t(1,p) = pR(t,1) + (1-p)R(t,0)$ is linear in p, there exists a unique $p_t^* \in (0,1)$ such that

$$V_t(p,1) > V_t(p,\emptyset) \text{ for all } p > p_t^*$$

$$V_t(p,1) < V_t(p,\emptyset) \text{ for all } p < p_t^*.$$

Thus, in equilibrium, the good agent's strategy must be such that

$$A_t^G(p) = \begin{cases} 0 & \text{for all } p < p_t^* \\ 1 & \text{for all } p > p_t^*. \end{cases}$$

Now, let us consider A_t^B. Proof by induction. Assume by induction that $A_s^B \in (0,1)$ for all $s < t$ (this holds vacuously when $t = 1$). Assume by contradiction $A_t^B \in \{0,1\}$. First, consider the case where $A_t^B = 0$. It follows from Bayes Rule that

$$R(t,0) = \frac{1}{1 + (\frac{1-R_{t-1}}{R_{t-1}})(\frac{Pr(\tau=t,\theta=0|\tau\notin\{1,...,t-1\},i=B)}{Pr(\tau=t,\theta=0|\tau\notin\{1,...,t-1\},i=G)})} \tag{5}$$

First, note that

$$Pr(\tau = t, \theta = 0 | \tau \notin \{1, ..., t-1\}, i = B) = A_t^B = 0.$$

Meanwhile,

$$Pr(\tau = t, \theta = 0 | \tau \notin \{1, ..., t-1\}, i = G) = \int_0^1 \int_{p_t^*}^1 (1 - p_t) dG_t(p_t | p_{t-1}) dH_{t-1}(p_{t-1}) > 0,$$

where the strict inequality follows from the fact that $p_t^* \in (0, 1)$. By Lemma 2, it follows from (18) that $R(t, 0) = 1$. One can analogously show that $R(t, 1) = 1$. Thus,

$$V_t(p_0, 1) = p_0 R(t, 1) + (1 - p_0) R(t, 0) = 1 \tag{6}$$

Now, by the Law of Iterated Expectations

$$\begin{aligned} R_{t-1} &= Pr(i = G, \tau = t | \tau \notin \{1, ..., t-1\})(1) \\ &+ Pr(i = G, \tau \neq t | \tau \notin \{1, ..., t-1\}) \int_0^1 V_t^G(\emptyset, p) dH_t(p_t) \\ &+ Pr(i = B | \tau \notin \{1, ..., t-1\}) V_t^B(p_0, \emptyset). \end{aligned} \tag{7}$$

Because R is consistent with the A^i in equilibrium, $V_t^B(p_0, \emptyset) \geq \int_0^1 V_t^G(p, \emptyset) dH_t(p_t)$. Because $R_{t-1} < 1$ (Lemma 2), it follows from (7) that $V_t^B(\emptyset, p_0) < 1$. Combining this with (6) implies $V_t^B(p_0, \emptyset) < V_t(p_0, 1)$. Thus, $A_t^B(p_0) = 1$. Contradiction. □

Proof of Claim 1. First, we want to show that in any equilibrium, $p^* = p_0$. Fix any equilibrium. By Proposition 1, $A_t^B \in (0, 1)$. Thus,

$$V(p_0, 1) = V^B(p_0, \emptyset) = V^G(p_0, \emptyset), \tag{8}$$

where the second equality follows from the fact that $T = 1$. Note further that (1) both $V(p, 1)$ and $V(p, \emptyset)$ are linear in p and (2) by (SC), $V(0, 1) < V^G(0, \emptyset)$ and $V(1, 1) > V^G(1, \emptyset)$. These two facts, combined with (8), imply that $V(p, 1) < V(p, \emptyset)$ for all $p < p_0$ and $V(p, 1) > V(p, \emptyset)$ for all $p > p_0$. Thus $p^* = p_0$.

Next, we want to show that there exists a unique $b \in (0, 1)$ such that $(A^B = b, p^* = p_0)$ is an equilibrium strategy. First, define

$$W(a, b) \equiv p_0 R^b(a, 1) + (1 - p_0) R^b(a, 0)$$

where

$$R^b(a,\theta) \equiv \frac{1}{1+\frac{1-R_0}{R_0}\frac{Pr(a,\theta|i=B,A^B=b)}{Pr(a,\theta|i=G,p^*=p_0)}}$$

is the unique reputation function that is consistent with the strategy profile $(A^B = b, p^* = p_0)$. I claim that there exists a unique $b \in (0,1)$ such that $W(1,b) = W(\emptyset, b)$. First, note that

$$Pr(a,\theta|i = G, p^* = p_0) \in (0,1) \text{ for all } a, \theta. \tag{9}$$

Now, I make two observations about W:

1. $W(1, b = 0) - W(\emptyset, b = 0) > 0$ and $W(1, b = 1) - W(\emptyset, b = 1) < 0$.
   To show this, note that $Pr(1,\theta|i = B, A^B = 0) = 0$ for all θ. Thus, by (9), $R^{b=0}(1,\theta) = 1$ and $R^{b=0}(\emptyset,\theta) < 1$ for all θ. Thus, $W(1, b = 0) - W(\emptyset, b = 0) > 0$. One can analogously show that $W(1, b = 1) - W(\emptyset, b = 1) < 0$.

2. $W(1, b) - W(\emptyset, b)$ is continuous and strictly decreasing in b.
   To show this, note that

   $$R^b(1,1) = \frac{1}{1+\frac{1-R_0}{R_0}\frac{p_0 b}{1-F(1|\theta=1)}},$$

   which is continuous and strictly decreasing in b. One can similarly show that $R^b(1,\theta)$ ($R^b(\emptyset,\theta)$) is continuous and strictly decreasing (increasing) in b for all θ. The statement then follows from the definition of W.

1. and 2. above imply that there exists a unique b such that $W(1, b) = W(\emptyset, b)$.

Finally, I claim that $(A^B = b, p^* = p_0)$ is the unique equilibrium strategy profile. Because $W(1, b) = W(\emptyset, b)$, $V(p_0, 1) = V(p_0, \emptyset)$ and thus $A^B = b$ is a best response. That $p^* = p_0$ is a best response follows from the fact that $V(p, 1) - V(p, \emptyset)$ is strictly increasing in p. Thus, we have shown $(A^B = b, p^* = p_0)$ is an equilibrium. It remains to show uniqueness. This follows from the fact that b is the unique value such that $W(1, b) = W(\emptyset, b)$, and thus the unique value such that $V(p_0, 1) = V(p_0, \emptyset)$ under the R that is consistent with this A^B. □

Proof of Claim 2. Fix an R_0 and $f(\cdot|\theta)$ for $\theta \in \{0,1\}$. Let b^1 (b^2) and R^1 (R^2) denote the equilibrium bad agent strategy and reputation function, respectively, under prior p_0^1 (p_0^2), where $p_0^1 < p_0^2$. We want to show that $b^1 < b^2$. Suppose by contradiction that $b^1 \geq b^2$. It follows from Bayes Rule and the good agent's strategy $p^* = p_0$ that for $k \in \{1,2\}$

$$R^k(1,1) = \frac{1}{1+\frac{1-R_0}{R_0}\frac{b^k}{1-F(1|\theta=1)}},$$

and thus $R^1(1,1) \leq R^2(1,1)$. One can analogously show that $R^1(1,0) \leq R^2(1,0)$, $R^1(\emptyset,1) \geq$

$R^2(\emptyset,1)$, and $R^1(\emptyset,0) \geq R^2(\emptyset,0)$. Further note that by the selection assumption,

$$\begin{aligned} V^k(1,1) > V^k(1,\emptyset) \Leftrightarrow R^k(1,1) > R^k(\emptyset,1) \\ V^k(0,1) > V^k(0,\emptyset) \Leftrightarrow R^k(1,0) < R^k(\emptyset,0) \end{aligned} \tag{10}$$

where V^k is the agents' equilibrium value function under p_0^k.

It follows from Proposition 1 that the bad agent must be indifferent between $a=1$ and $a=\emptyset$ at p_0. This implies

$$\frac{p_0^k}{1-p_0^k} = \frac{R^k(\emptyset,0) - R^k(1,0)}{R^k(1,1) - R^k(\emptyset,1)} \text{ for } k \in \{1,2\}. \tag{11}$$

But it follows from the above inequalities and (10) that if (11) holds for $k=1$, then it fails for $k=2$, namely

$$\frac{p_0^2}{1-p_0^2} > \frac{R^2(\emptyset,0) - R^2(1,0)}{R^2(1,1) - R^2(\emptyset,1)}.$$

Contradiction. □

We now seek to establish existence of an equilibrium. To this end, let us define the correspondence Φ as follows. First, let us define R^x. Let R^x denote the reputation function that is consistent with the strategy profile $x = (p_1^*, ..., p_T^*, A_1^B, ..., A_T^B) \in [p_0, 1]^T \times [0,1]^T$. Formally, whenever Bayes Rule applies, $R^x(t,\theta)$ is given by

$$R^x(t,\theta) = \frac{1}{1 + (\frac{1-R_{t-1}}{R_{t-1}})(\frac{Pr(\tau=t,\theta|\tau\notin\{1,...,t-1\},i=B)}{Pr(\tau=t,\theta|\tau\notin\{1,...,t-1\},i=G)})}, \tag{12}$$

where the probabilities, including R_{t-1}, are those that obtain given the strategy profile x. The only case in which Bayes Rule does not apply is when $p_t^* = 1$ and $A_t^B = 0$ for some t, and in this case we impose $R^x(t,\theta) = 1$ for all θ.

Now, let $V_{s-1}^{G,x}(p, (\hat{p}_t)_{t=s}^T)$ denote G's value, under belief p at time $s-1$, from playing cutoff strategies $(\hat{p}_t)_{t=s}^T$ in periods $s, ..., T$, respectively, given reputation function R^x and that the agent did not act in $s-1$. Now, define the $\Phi_s^G(x)$ recursively as follows:

$$\Phi_s^G(x) \equiv \min_{\overline{p}_s \in [p_0,1]} \arg\max_{\overline{p}_s \in [p_0,1]} [V_{s-1}^{G,x}(p_0, (\overline{p}_t)_{t=s}^T)],$$

where $\overline{p}_t \equiv \Phi_t^G(x)$ for all $t > s$. Let $\Phi^G(x) \equiv (\Phi_t^G(x))_{t=1}^T$. Note that the value could have been taken at any interior belief (not necessarily p_0) and the analysis that follows would remain unchanged.

Next, let $V_s^{B,x}((b_t)_{t=s}^T)$ denote B's value from playing strategy $A_t^B = b_t$ for all $t \geq s$, given reputation function R^x and that the agent did not act before s. Now, define the $\Phi_s^B(x)$ recursively as follows:

$$\Phi_s^B(x) \equiv \arg\max_{b_s \in [0,1]} V_s^{B,x}((b_t)_{t=s}^T),$$

where $b_t \in \Phi_t^B(x)$ for all $t > s$. Define $\Phi^B(x) \equiv \Phi_1^B(x) \times ... \times \Phi_T^B(x)$, and finally $\Phi(x) \equiv \Phi^G(x) \times \Phi^B(x)$.

I wish to show that any fixed point of Φ is an equilibrium that satisfies (SC). To this end, I begin by establishing two lemmas.

Lemma 3. *In any fixed point of* Φ, $A_t^B \in (0,1)$ *and* $p_t^* < 1$ *for all* t.

Proof. Fix a $t \in \{1, ..., T\}$. Suppose by induction that $b_s \in (0,1)$ and $p_s^* < 1$ for all $s < t$. This holds vacuously for $t = 1$. Let R_{t-1}^x denote the interim reputation given reputation function R^x, which is given by (3). The inductive assumption implies that $R_{t-1}^x \in (0,1)$.

First, I show that $A_t^B \neq 0$. Suppose by contradiction that $A_t^B = 0$. If $p_t^* < 1$, it follows from (12) that $R^x(t,\theta) = 1$ for all θ. If $p_t^* = 1$, it follows by definition that $R^x(t,\theta) = 1$ for all θ. Thus, $V_t^{B,x}((b_s)_{s=t}^T) = 1$ for $b_t = 1$. Meanwhile, because $R_{t-1}^x \in (0,1)$ and $p_t^* \geq p_0$, it must be that $V_t^{B,x}((b_s)_{s=t}^T) < 1$ for $b_t = A_t^B$. Since $A_t^B = 0$, $A_t^B \notin \Phi_t^B(x)$, and hence x is not a fixed point. Contradiction.

Next, I show $A_t^B \neq 1$. Suppose by contradiction that $A_t^B = 1$. It follows from (12) that $R^x(t,\theta) < 1$ and $R^x(s,\theta) = 1$ for all θ and $s > t$. Thus,

$$V_t^{B,x}((b_s)_{s=t}^T) = 1 \text{ for } b_t = 0, \text{ and}$$

$$V_t^{B,x}((b_s)_{s=t}^T) < 1 \text{ for } b_t = A_t^B,$$

where the second statement follows from $R_{t-1}^x \in (0,1)$, and the Martingale property of the belief about i. Since $A_t^B = 1$, $A_t^B \notin \Phi_t^B(x)$. Contradiction.

Finally, I show that $p_t^* < 1$. Suppose by contradiction that $p_t^* = 1$. I showed above that $A_t^B \in (0,1)$. So, by (12), $R(t,\theta) = 0$ for all θ. By the Martingale property of the belief on i, $R(s,\theta) > 0$ for some $s \in \{t+1, ..., T, \emptyset\}$. Thus, $V_t^{B,x}((b_s)_{s=t}^T) < V_t^{B,x}((\tilde{b}_s)_{s=t}^T)$ for $b_s = A_s^B$ for all $s \geq t$ and $\tilde{b}_t = 0$, $\tilde{b}_s = A_s^B$ for all $s > t$. Thus, $A_t^B \notin \Phi_t^B(x)$. Contradiction. $\square$

Lemma 4. *For any fixed point* x *of* Φ*:*

1. $R^x(t,1) > R^x(t+1,1)$ *for all* $t < T$,
2. $R^x(t,0) < R^x(\emptyset,0)$ *and* $R^x(t,1) > R^x(\emptyset,0)$ *for all* $t \in \{1, ..., T\}$.

Proof. Let us begin by showing 2. By the same reasoning as that which is presented in Proposition 3,

$$R^x(t, \theta = 1) > R^x(t, \theta = 0) \text{ for all } t \in \{1, ..., T\} \text{ and } R^x(\emptyset, \theta = 1) < R^x(\emptyset, \theta = 0). \quad (13)$$

Because by Lemma 3, $A_t^B \in (0, 1)$ for all t,

$$p_0 R^x(t, 1) + (1 - p_0) R^x(t, 0) = p_0 R^x(\emptyset, 1) + (1 - p_0) R^x(\emptyset, 0).$$

This, together with (13), implies 2.

Now, let us show 1. Suppose by contradiction that there exists t such that

$$R^x(t, 1) \leq R^x(t + 1, 1).$$

This, combined with 2, implies that

$$V_{t-1}^{G,x}(p_0, (\overline{p}_s)_{s=t}^T) > V_{t-1}^{G,x}(p_0, (p_s^*)_{s=t}^T),$$

where $\overline{p}_t = 1$, $\overline{p}_s = p_s^*$ for all $s > t$. Thus, $p_t^* \neq \Phi_t^G(x)$. Contradiction. □

We are now ready to show that any fixed point of Φ is an equilibrium. This is formalized as Lemma 5.

Lemma 5. *Any fixed point x of Φ, together with R^x, is an equilibrium that satisfies (SC).*

Proof. Fix any fixed point x of Φ. First, I show that (x, R^x) satisfies (SC). Let V denote the value function given reputation function R^x and strategy profile x. It follows from Lemma 3 and Lemma 4 that for all t:

$$R^x(t, 1) = V_t^G(1, p = 1) = R^x(t, 1) > R^x(s, 1) = V_t^G(\emptyset, p = 1) \text{ for } s \in \{\emptyset, t + 1\}, \text{ and}$$

$$V_t^G(\emptyset, p = 0) = R^x(\emptyset, 0) > R^x(t, 0) = V_t^G(1, p = 0).$$

Thus, (SC) is satisfied.

It remains to show that (x, R^x) is an equilibrium. It follows from the definition of R^x that R^x is consistent with Bayes' Rule, given x. Next, I will show that given R^x, $(p_t^*)_{t=1}^T$ and $(A_t^B)_{t=1}^T$ are optimal for G and B, respectively. Since x is a fixed point, $A_t^B \in \Phi_t^B(x)$ for all x and the optimality of A_t^B follows from the definition of Φ_t^B. Next, consider G. By the same reasoning as presented in the proof of Proposition 2, given that (SC) is satisfied for all t, there exists a $\hat{p}_t \in (0,1)$ such that the unique optimal strategy is the cutoff strategy $\hat{p}_t$ (given R^x). It remains to show that for all t, $\hat{p}_t = p_t^*$. Fix a t and suppose by induction that $\hat{p}_s = p_s^*$ for all $s > t$. By the definition of Φ_t^G, it follows that $\Phi_t^G(x) = \hat{p}_t$.

□

Proof of Proposition 2: existence. I now establish existence of a fixed point to Φ. It follows from Lemma 5 that this is an equilibrium. To this end, for each $\varepsilon > 0$, I define a constrained correspondence Φ^ε and show that for some ε, there exists a fixed point of Φ^ε which is also a fixed point of Φ. I proceed in a number of steps, as outlined below.

1. **Define constrained correspondence:** For any $\varepsilon \in (0, 1-p_0)$, let Φ^ε be identical to Φ, except that the domain and range are constrained as follows:

$$\Phi^\varepsilon : [p_0, 1-\varepsilon]^T \times [0,1]^T \to [p_0, 1-\varepsilon]^T \times (2^{[0,1]})^T.$$

Now, define

$$\Phi_s^{G,\varepsilon}(x) \equiv \min_{\overline{p}_s \in [p_0, 1-\varepsilon]} \arg\max_{\overline{p}_s \in [p_0, 1-\varepsilon]} [V_{s-1}^{G,x}(p_0, (\overline{p}_t)_{t=s}^T)], \tag{14}$$

and let $\Phi^\varepsilon(x) \equiv \Phi^{G,\varepsilon}(x) \times \Phi^B(x)$, where $\Phi^B(x)$ is defined as before.

2. **Existence of fixed point for Φ^ε:** I now claim that for any $\varepsilon < 1 - p_0$, Φ^ε has a fixed point. To prove this, I invoke the Kakutani fixed point theorem. To this end, I show that Φ^ε satisfies the following properties:

   (a) $\Phi^\varepsilon(x)$ **is non-empty for all** x**.** This follows from the fact that $[p_0, 1-\varepsilon]$ and $[0,1]$ are compact and $R^x(\tau, \theta)$ is bounded for all (τ, θ), implying by the Extreme Value Theorem that both $\Phi_t^B(x)$ and $\Phi_t^{G,\varepsilon}(x)$ are non-empty for all t, x.

   (b) $\Phi^\varepsilon(x)$ **is convex and closed for all** x**.** $\Phi_t^{G,\varepsilon}(x)$ is a singleton by definition for all x, t. Now, fix an (x,t) and consider $\Phi_t^B(x)$. Now, define $\underline{b}_t = 0$, $\overline{b}_t = 1$, $\overline{b}_s = \underline{b}_s = A_s^B$ for all $s > t$. It follows that

$$\Phi_t^B(x) = \begin{cases} 1 & \text{if } V_t^{B,x}((\underline{b}_s)_{s=t}^T) < V_t^{B,x}((\overline{b}_s)_{s=t}^T) \\ 0 & \text{if } V_t^{B,x}((\underline{b}_s)_{s=t}^T) > V_t^{B,x}((\overline{b}_s)_{s=t}^T) \\ [0,1] & \text{if } V_t^{B,x}((\underline{b}_s)_{s=t}^T) = V_t^{B,x}((\overline{b}_s)_{s=t}^T), \end{cases} \tag{15}$$

   and thus $\Phi_t^B(x)$ is convex and closed. It follows that $\Phi^\varepsilon(x)$ is also convex and closed.

   (c) Φ^ε **is upper hemi-continuous (UHC).** I will show that for all t, Φ_t^B and $\Phi_t^{G,\varepsilon}$ are

UHC everywhere on the domain. It follows that their Cartesian product Φ^ε is also UHC. Fix an $x \in X$ and a t. Let us begin with Φ_t^B. Now note that because $\varepsilon > 0$, $R^x(t,\theta)$ is continuous in x, and thus both $V_t^{B,x}((\underline{b}_s)_{s=t}^T)$ and $V_t^{B,x}((\bar{b}_s)_{s=t}^T)$ are continuous in x. Thus, it follows from (15) that $\Phi_t^B(x)$ is UHC at x. Next, consider $\Phi_t^{G,\varepsilon}$. It again follows from the continuity of $R^x(t,\theta)$ that $V_t^{G,x}$ is continuous in x, and thus by (14), $\Phi_s^{G,\varepsilon}(x)$ is continuous in x.

It follows then from the Kakutani fixed point theorem that Φ^ε has a fixed point.

3. **Show that for some $\varepsilon > 0$, Φ^ε has an interior fixed point:** I now claim that for some $\varepsilon > 0$, Φ^ε has a fixed point that lies within $[p_0, 1-\varepsilon)^T \times [0,1]^T$ (i.e., a fixed point such that $p_t^* < 1-\varepsilon$ for all t). Suppose not, by contradiction. Then, there exists $t^* < T$ and a sequence $\{\varepsilon_n\}_{n=1}^\infty$ such that $\varepsilon_n > 0$ for all n, $\lim_{n\to\infty} \varepsilon_n = 0$ and there exists a sequence $\{x_n\}_{n=1}^\infty$ where x_n is a fixed point of Φ^{ε_n} such that $p_{t^*}^* = 1-\varepsilon_n$.

   I now claim that

$$\lim_{n\to\infty} R^{x_n}(s,1) - R^{x_n}(s+1,1) = 0 \text{ and } \lim_{n\to\infty} R^{x_n}(s,0) = 0 \tag{16}$$

   for all $s \geq t^*$. Proof by induction. Begin with $s = t^*$. Note that by the contradiction assumption, for all n, $V_{t^*}^G(1, 1-\varepsilon_n) \leq V_{t^*}^G(\emptyset, 1-\varepsilon_n)$ (where this is the value function that obtains from R^{x_n}) because otherwise $p_t^* < 1-\varepsilon_n$ under x_n. I claim this implies $\lim_{n\to\infty} R^{x_n}(t^*,1) - R^{x_n}(t^*+1,1) = 0$. Suppose not, by contradiction. Then there exists $\delta > 0$ and an infinite subsequence $\{\varepsilon_{n_k}\}_{k=1}^\infty$ of $\{\varepsilon_n\}_{n=1}^\infty$ where $n_1 < n_2 < \ldots \in \mathbb{N}$ such that $R^{x_{n_k}}(t^*,1) - R^{x_{n_k}}(t^*+1,1) > \delta$ for all k. Thus, there exists k such that $V_{t^*}^G(1, 1-\varepsilon_{n_k}) - V_{t^*}^G(\emptyset, 1-\varepsilon_{n_k}) > 0$. Contradiction.

   Next, I show $\lim_{n\to\infty} R^{x_n}(t^*,0) = 0$. Recall that by Bayes Rule, under any x_n:

$$R^{x_n}(t^*,0) = \frac{1}{1 + (\frac{1-R_0}{R_0})(\frac{1}{Q_{t^*}(n)})} \text{ where } Q_t(n) \equiv \frac{Pr(\theta=0|\tau=t, i=G)Pr(\tau=t|i=G)}{Pr(\theta=0|\tau=t, i=B)Pr(\tau=t|i=B)},$$

   and the probabilities are those that obtain under the strategy profile x_n. I claim that $\lim_{n\to\infty} Q_{t^*}(n) = 0$. Suppose not, by contradiction. Since $\lim_{n\to\infty} \varepsilon_n = 0$, $\lim_{n\to\infty} \frac{Pr(\theta=0|\tau=t,i=G)}{Pr(\theta=0|\tau=t,i=B)} = 0$, and thus it suffices to show that $\frac{Pr(\tau=t^*|i=G)}{Pr(\tau=t^*|i=B)}$ does not diverge as $n \to \infty$. This is only possible if there exists a subsequence $\{\varepsilon_{n_k}\}_{k=1}^\infty$ of $\{\varepsilon_n\}_{n=1}^\infty$ such that $\lim_{k\to\infty} \frac{Pr(\tau=t^*|i=G)}{Pr(\tau=t^*|i=B)} = \infty$. This implies $\lim_{k\to\infty} R^{x_{n_k}}(t^*,\theta) = 1$ for all θ, and thus for k sufficiently large, $A_{t^*}^B = 1$ is a profitable deviation from what is specified under x_{n_k}. Thus, x_{n_k} is not a fixed point of $\Phi^{\varepsilon_{n_k}}$. Contradiction.

   Now, fix some $t > t^*$ and assume by induction that (16) holds for all s such that

$t^* \leq s < t$. We want to show that it also holds for t. First, let us show that $\lim_{n\to\infty} R^{x_n}(t,0) = 0$. For all n, because x_n is a fixed point, $A_t^B \in (0,1)$, and thus

$$p_0 R^{x_n}(t,1) + (1-p_0)R^{x_n}(t,0) = p_0 R^{x_n}(t-1,1) + (1-p_0)R^{x_n}(t-1,0).$$

Thus,

$$\lim_{n\to\infty} R^{x_n}(t,0) = \frac{p_0}{1-p_0}[\lim_{n\to\infty}[R^{x_n}(t-1,1) - R^{x_n}(t,1)] - \lim_{n\to\infty} R^{x_n}(t-1,0)] = 0,$$

where the last equality follows from the inductive assumption.

Next, let us show that $\lim_{n\to\infty} R^{x_n}(t,1) - R^{x_n}(t+1,1) = 0$. Suppose not, by contradiction. Then, there exists $\delta > 0$ and subsequence $\{\varepsilon_{n_k}\}_{k=1}^{\infty}$ of $\{\varepsilon_n\}_{n=1}^{\infty}$ such that $R^{x_{n_k}}(t,1) - R^{x_{n_k}}(t+1,1) \geq \delta$ for all k. This implies that there exists $p \in (p_0,1)$ such that $p_t^* \leq p$ under x_{n_k} for all k. However, $\lim_{n\to\infty} R^{x_n}(t,0) = 0$, and thus for all $p \in (p_0,1)$, there exits an $N \in \mathbb{N}$ such that $p_t^* > p$ under x_n for all $n > N$. Contradiction.

Now, note that for all n, $p_t^* = p_0$ under x_n (this follows from identical reasoning to that presented in the proof of Claim 1). Thus, $Q_T(n)$ does not converge to 0 as $n \to \infty$. However, because $\lim_{n\to\infty} R^x(T,0) = 0$, $\lim_{n\to\infty} Q_T(n) = 0$. Contradiction.

4. **This interior fixed point of Φ^ε is also a fixed point of Φ:** Fix an $\varepsilon > 0$ such that there is a fixed point x of Φ^ε such that $x \in [p_0, 1-\varepsilon)^T \times [0,1]^T$. I claim that x is also a fixed point of Φ. This is equivalent to showing that for all t:

$$A_t^B \in \Phi_t^B(x) \text{ and } p_t^* = \Phi_t^G(x).$$

Note that $A_t^B \in \Phi_t^B(x)$ holds because this is necessary for x to be a fixed point of Φ^ε. Next, let us show that $p_t^* = \Phi_t^G(x)$ for all t. Proof by induction. Fix a t, and suppose $p_s^* = \Phi_s^G(x)$ for all $s > t$. We want to show $p_t^* = \Phi_t^G(x)$.

By the same reasoning that is presented in the proof of Proposition 1, since $p_t^* < 1-\varepsilon$,

$$V_t^G(p,1) > V_t^G(p,\emptyset) \text{ for all } p > 1-\varepsilon.$$

where this is the value function that obtains given the reputation function R^x. Thus,

$$V_t^{G,x}(p_0, (p_s^*)_{s=t}^T) > V_t^{G,x}(p_0, (\tilde{p}_s)_{s=t}^T)$$

for any $\tilde{p}_t > 1-\varepsilon$ and $\tilde{p}_s = p_t^*$ for all $s > t$. This, combined with the fact that $p_t^* = \Phi_t^{G,\varepsilon}(x)$, implies $p_t^* = \Phi_t^G(x)$.

$\square$

Proof of Proposition 2: cutoff bounds. Consider any equilibrium that satisifes (SC). First, want to show that $p_T^* = p_0$. It follows from Lemma 1 and Proposition 1 that $R_{T-1} \in (0,1)$. Thus, by the same reasoning presented in Claim 1, $p_T^* = p_0$.

Now, want to show that for all $t < T$, $p_t^* > p_0$. To this end, fix a $t < T$. It follows from Proposition 1 that B mixes between $a \in \{1, \emptyset\}$ in every t, and thus

$$V_t(p_0, 1) = V_t^B(p_0, \emptyset) \quad \text{and} \quad V_t(p_0, 1) = V_t^B(p_0, \emptyset).$$

So, $V_t(p_0, 1) = V_{t+1}(p_0, 1)$. Now it follows from (SC) that

$$V_t(1, \emptyset) = V_{t+1}(1, 1) \quad \text{and} \quad V_t(0, \emptyset) > V_{t+1}(0, 1).$$

By Lemma 1, it follows that $V_t(p, \emptyset) > V_{t+1}(p, 1)$ for all $p < 1$. Because $p_0 \in (0,1)$, then

$$V_t(p_0, \emptyset) > V_{t+1}(p_0, 1) \tag{17}$$

Finally, it follows from the same reasoning presented in the proof of Proposition 1 that $V_t^G(p, \emptyset) > V_t(p, 1)$ if and only if $p < p_t^*$. Thus, it follows from (17) that $p_t^* > p_0$.

$\square$

Proof of Proposition 3. Let us begin by showing that $R(t, \theta = 1) > R(t, \theta = 0)$ for all $t \in \{1, ..., T\}$. To this end, note that

$$R(t, \theta) = \frac{1}{1 + (\frac{1-R_{t-1}}{R_{t-1}})(\frac{Pr(\tau=t,\theta|\tau \notin \{1,...,t-1\}, i=B)}{Pr(\tau=t,\theta|\tau \notin \{1,...,t-1\}, i=G)})}. \tag{18}$$

Now, note the following:

- $Pr(\tau = t, \theta = 0 | \tau \notin \{1, ..., t-1\}, i = B) = (1 - p_0)A_t^B$
- $Pr(\tau = t, \theta = 1 | \tau \notin \{1, ..., t-1\}, i = B) = p_0 A_t^B$
- $Pr(\tau = t, \theta = 0 | \tau \notin \{1, ..., t-1\}, i = G) = \int_0^1 \int_{p_t^*}^1 (1 - p_t) dG_t(p_t|p_{t-1}) dH_{t-1}(p_{t-1})$
- $Pr(\tau = t, \theta = 1 | \tau \notin \{1, ..., t-1\}, i = G) = \int_0^1 \int_{p_t^*}^1 p_t dG_t(p_t|p_{t-1}) dH_{t-1}(p_{t-1})$.

Thus,

$$R(t, 0) = \frac{1}{1 + (\frac{1-R_{t-1}}{R_{t-1}})(\frac{A_t^B}{\int_0^1 \int_{p_t^*}^1 \frac{1-p_t}{1-p_0} dG_t(p_t|p_{t-1}) dH_{t-1}(p_{t-1})})}$$

$$R(t,1) = \frac{1}{1 + (\frac{1-R_{t-1}}{R_{t-1}})(\frac{A_t^B}{\int_0^1 \int_{p_t^*}^1 \frac{p_t}{p_0} dG_t(p_t|p_{t-1}) dH_{t-1}(p_{t-1})})}.$$

Now, define $X(p) \equiv \frac{1-p}{1-p_0}$ and $Y(p) \equiv \frac{p}{p_0}$. Note that

$$X(1) = 0, X(p_t^*) \leq 1, X(p) \text{ is strictly decreasing in } p$$

$$Y(1) > 1, Y(p_t^*) \geq 1, Y(p) \text{ is strictly increasing in } p,$$

where the inequalities follow from Proposition 2. This implies that $Y(p) > X(p)$ for all $p \in (p_t^*, 1]$. Thus, $R(t,0) < R(t,1)$.

It remains to show that $R(\emptyset, 1) < R(\emptyset, 0)$. Note that

$$R(\emptyset,0) = \frac{1}{1 + (\frac{1-R_{T-1}}{R_{T-1}})(\frac{1-A_t^B}{\int_0^1 \int_0^{p_0} X(p_T) dG_T(p_T|p_{T-1}) dH_{T-1}(p_{T-1})})}$$

$$R(\emptyset,1) = \frac{1}{1 + (\frac{1-R_{T-1}}{R_{T-1}})(\frac{1-A_t^B}{\int_0^1 \int_0^{p_0} Y(p_T) dG_T(p_T|p_{T-1}) dH_{T-1}(p_{T-1})})}.$$

Now note

$$X(0) > 1, X(p_0) = 1, Y(0) = 0, Y(p_0) = 0.$$

These facts, combined with the monotonicity of X and Y in p implies that $X(p) > Y(p)$ for all $p \in [0, p_0)$. Thus, $R(\emptyset, 0) > R(\emptyset, 1)$.

□

Proof of Proposition 4. Fix any $t < T$. We want to show that

$$R(t, \theta = 1) > R(t+1, \theta = 1) \text{ and } R(t, \theta = 0) < R(t, \theta = 0).$$

First, note that $G(p_{t+1}|p_t = 1) = F(0) = 1$. Thus, since $p_{t+1}^* \in (0,1)$,

$$V_t(1,\emptyset) = \int_0^{p_{t+1}^*} V_{t+1}(p_{t+1},\emptyset) dG_t(p_{t+1}|p_t = 1) + \int_{p_{t+1}^*}^1 V_{t+1}(p_{t+1},1) dG_t(p_{t+1}|p_t = 1) = V_{t+1}(1,1).$$

So,

$$V_t(1,1) = V_{t+1}(1,1) \Leftrightarrow R(t,1) = R(t+1,1). \tag{19}$$

Next, recall that by the bad agent's indifference condition,

$$V_t(1,p_0) = V_{t+1}(1,p_0) \Leftrightarrow p_0 R(t,1) + (1-p_0)R(t,\theta=0) = p_0 R(t+1,1) + (1-p_0)R(t+1,0)$$

This, combined with (19), implies that $R(t,0) < R(t+1,0)$. □

In what follows, the two input arguments on the value function have been reversed, i.e., the value is given by $V_t(a,p)$ rather than $V_t(p,a)$.

Proof of Proposition 5. Part 1 follows from the fact that the B agent holds belief p_0 in every period. For part 2, let us first show that G plays a cutoff rule and these cutoffs are unique. Proof by induction.

Base case: In period T, the agent acts if and only if his belief lies above $p^* \equiv \frac{-K_0}{K_1-K_0}$.

Induction step: Fix a t and suppose that the agent plays an interior cutoff rule in all $s < t$. Note that

$$V_t(1,p) = \beta^t(pK_1 + (1-p)K_0).$$

Now, let us observe three facts about $V_t^G(\emptyset,p)$:

1. Since a cutoff rule is played in $t+1$,

$$V_t^G(\emptyset,1) = V_{t+1}^G(1,1) = \beta^{t+1}K_1 < \beta^t K_1 = V_t^G(1,1)$$

2. $V_t^G(\emptyset,0) = 0 > \beta^t K_0 = V_t^G(1,0)$

3. $V_t(\emptyset,p)$ is convex in p.

These three facts together with the linearity of $V_t(1,p)$ imply that there is a unique $p_t^* \in (0,1)$ such that (1) holds.

Now, it remains to show that the p_t^* are strictly decreasing in t. To this end, fix a $t < T$. Suppose by contradiction that $p_t^* \le p_{t+1}^*$. Then

$$V_t^G(1,p_t^*) = V_t^G(\emptyset,p_t^*)$$

$$V_{t+1}^G(1,p_t^*) \le V_{t+1}^G(\emptyset,p_t^*)$$

Since all these values are strictly positive

$$\beta = \frac{V_{t+1}^G(1,p_t^*)}{V_t^G(1,p_t^*)} \le \frac{V_{t+1}^G(\emptyset,p_t^*)}{V_t^G(\emptyset,p_t^*)}. \tag{20}$$

Now, let $\tilde{V}_t(\emptyset, p)$ denote the agent's value from the modified problem which is identical to the original problem except that the time horizon is $T-1$. It follows that for all $t < T$:

1. $\tilde{V}_t^G(\emptyset, p) = \frac{V_{t+1}^G(\emptyset,p)}{\beta}$
2. $\tilde{V}_t^G(\emptyset, p) < V_t^G(\emptyset, p)$.

These two facts together imply

$$\frac{V_{t+1}^G(\emptyset, p_t^*)}{V_t^G(\emptyset, p_t^*)} < \beta,$$

contradicting (20). □

Proof of Proposition 6. Define $\underline{K} \equiv [\frac{-X}{(1-X)\beta^T} - K_1 p_0](\frac{1}{1-p_0})$.

First, I show that in any equilibrium where $K_0 < \underline{K}$, $A_t^B = 0$ for all t. Note that in any equilibrium, for any t,

$$V_t^B(1, p_0) \leq \beta^t(1-X)[K_1 p_0 + K_0(1-p_0)] + X$$

$$V_t^B(\emptyset, p_0) \geq 0.$$

Furthermore, when $K_0 < \underline{K}$, $V_t(1, p_0) < 0$. This implies that $V_t^B(1, p_0) < V_t^B(\emptyset, p_0)$, and thus $A_t^B = 0$.

Now, want to show that for all t, $\hat{p}_t > p_t^*$. Because G plays a cutoff strategy in equilibrium, it suffices to show that

$$V_t^G(1, \hat{p}_t) > V_t^G(\emptyset, \hat{p}_t) \text{ for all } t.$$

Now, let $\hat{V}$ denote the value function under the no-reputation benchmark ($X = 0$). For all t and p:

1. $V_t^{NR,G}(1, p) = \hat{V}_t^G(1, p)$
2. $V_t^{NR,G}(\emptyset, p) \leq \hat{V}_t^G(\emptyset, p)$
3. $V_t^{R,G}(1, p) > V_t^{R,G}(\emptyset, p)$,

where the final inequality follows from the fact that $A_t^B = 0$ and $p_t^* \in (0, 1)$ for all t, and thus in equilibrium $V_t^{R,G}(1, p) = 1$ whereas $V_t^{R,G}(\emptyset, p) < 1$. Furthermore, by definition of $\hat{p}_t$, $\hat{V}_t^G(1, \hat{p}_t) = \hat{V}_t^G(\emptyset, \hat{p}_t)$ for all t. It follows from the above facts that for all t:

$$V_t(1, \hat{p}_t) = (1-X)V_t^{NR}(1, \hat{p}_t) + XV_t^R(1, \hat{p}_t) > (1-X)V_t^{NR}(\emptyset, \hat{p}_t) + XV_t^R(\emptyset, \hat{p}_t) = V_t(\emptyset, \hat{p}_t),$$

and thus $p_t^* < \hat{p}_t$. □

Proof of Proposition 7. Fix any equilibrium $(p_t^*, A_t^B)_{t=1}^T$ and any $t \in \{1,..,T\}$. First, consider the case where $A_s^B = 1$ for some $s < t$. Since G plays an interior cutoff strategy at all t, the equilibrium reputation function R must be such that

$$R(\tau, \theta) = 1 \text{ for all } \tau \in \{t,..,T,\emptyset\}, \theta \in \{0,1\}.$$

Thus, $V_t^R, G = 1$ for all $a \in \{\emptyset, 1\}$, $p \in [0,1]$. Hence, G's problem at time t is to choose a strategy which maximizes the following:

$$E_\theta[U(\tau,\theta)] = (1-X)\beta^\tau(K_\theta \mathbb{I}(\tau \neq \emptyset)) + X.$$

This problem is equivalent to maximizing $\beta^\tau(K_\theta \mathbb{I}(\tau \neq \emptyset))$. Hence, the equilibrium strategy must be equal to the optimal cutoff rule under $X = 0$, i.e., $p_t = \hat{p}_t$.

Next, consider the case where $A_s^B < 1$ for all $s < t$. I claim that in this case $p_t^* > \hat{p}_t$. First, suppose that $A_t^B = 1$. The equilibrium reputation function must be such that (1) $R(s,\theta) = 1$ for all $s \in \{t+1,...,T,\emptyset\}$ and $\theta \in \{0,1\}$ and (2) $R(t,\theta) < 1$ for $\theta \in \{0,1\}$. Together, these two facts imply that $V_t^{R,G}(1,p) < 1$ and $V_t^{R,G}(\emptyset,p) = 1$. for all p. Furthermore, by the same reasoning as above, $p_s^* = \hat{p}_s$ for all $s > t$ and thus $V_t^{NR,G}(a,p) = \hat{V}_t(a,p)$ for all a, p. Thus,

$$V_t^G(1,\hat{p}_t) = (1-X)V_t^{NR,G}(1,\hat{p}_t) + XV_t^{R,G}(1,\hat{p}_t) < (1-X)V_t^{NR,G}(\emptyset,\hat{p}_t) + XV_t^{R,G}(\emptyset,\hat{p}_t) = V_t^G(\emptyset,\hat{p}_t),$$

and thus $p_t^* > \hat{p}_t$. Next, suppose that $A_t^B < 1$. It must also be that $A_t^B > 0$. To show this, suppose not by contradiction. Then, the reputation function must be such that $R(t,\theta) = 1$ for $\theta \in \{0,1\}$. Thus, $V_t^{R,B}(1,p) \geq V_t^{R,B}(\emptyset,p)$ for all p. Since $V_t^{NR,B}(1,p_0) > V_t^{NR,B}(\emptyset,p_0)$, it follows that $V_t^B(1,p_0) > V_t^B(\emptyset,p_0)$, and thus $A_t^B = 1$. Contradiction. So, $A_t^B \in (0,1)$ which implies B must be indifferent at p_0: $V_t^B(1,p_0) = V_t^B(\emptyset,p_0)$. Since $V_t^G(\emptyset,p_0) \geq V_t^B(\emptyset,p_0)$ and $V_t^B(1,p_0) = V_t^G(1,p_0) \leq V_t^G(\emptyset,p_0)$, $V_t^G(\emptyset,p_0) \geq V_t^G(1,p_0)$. It follows that $p_t^* \geq p_0 > \hat{p} \geq \hat{p}_t$.

□