Macro Qualifier Exam

Penn State: August, 2024

- You have 3.5 hours. There are four (multi-part) questions, one for each quarter of the first-year macro sequence. Each question is worth 45 points, so if you progress at a rate of one point per minute, you will be able to complete the exam with some time to spare.
- Neither books nor notes are permitted.
- If you make any assumptions beyond what's in the text of the question, please state those assumptions clearly.
- If you need more space, please ask for additional sheets of paper. If you use more sheets, please number the pages, write your identifying number instead of your name, and label clearly which question you are answering.
- Please write clearly. Show intermediate work for partial credit. Illegible work will receive no credit.

Good luck!

Question 1. An OLG Model with Production

Consider a two-period overlapping generation model of Diamond (1965) with production. The population of each generation is constant N. Agents in each generation are identical: each has 1 unit of labor when young, which he supplies to the labor market inelastically. He can not work when old. A young agent saves part of his labor earnings, which turns into an equal amount of capital when he is old. He supplies his capital holdings to the capital market and earns rent. Each initial old agent is endowed with $k_1 > 0$ units of capital at $t = 1$ and consumes the income from his capital endowment. A generation-t agent's preference over two-period consumption, c_t^t when young and c_{t+1}^t when old, is given by

$$u(c_t^t, c_{t+1}^t) = \ln(c_t^t) + \beta \ln(c_{t+1}^t)$$

where $\beta \in (0, 1)$. The production function in an arbitrary period t is given by

$$F(K_t, L_t) = K_t^\alpha L_t^{1-\alpha}$$

where $\alpha \in (0, 1)$, and K_t and L_t are the capital and labor employed in production at t, respectively. Capital depreciates fully each period after being used. All markets (goods, capital, and labor) in each period are perfectly competitive.

1. (14 Points) Competitive equilibrium.
   (a) Formulate the individual optimization problem of a representative generation-t agent and derive the agent's optimal saving function.
   (b) Define a competitive equilibrium. Explicitly specify the set of equations that a competitive equilibrium must satisfy.
2. (21 Points) A steady-state equilibrium (SSE) is a competitive equilibrium where the allocation is stationary over time.
   (a) Solve explicitly an SSE with positive production in terms of the model parameters. At this equilibrium, denote the capital per unit of labor by k^*.
   (b) Is this SSE locally stable? If you need additional conditions to determine the local stability of this SSE, state the conditions. Show your argument.
   (c) Is this SSE Pareto efficient? What are the conditions on the model parameters such that the SSE is Pareto optimal? Show your argument.
3. (10 Points) Suppose the conditions in (2.b) (if needed) are satisfied, but conditions in (2.c) do not hold, i.e., the SSE is locally stable but not Pareto efficient. Furthermore, assume that $k_1 \neq k^*$. Can an infinitely-lived benevolent government correct the inefficiency by implementing some government policy? If the answer is yes, outline the policy. If the answer is no, explain why.

Question 2. A Consumption-Saving Problem

Consider an economy populated by a continuum of infinitely-lived agents with measure 1. There is one perishable endowment/consumption good per period. Each agent's preference over sequences of consumption is given by

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where $\beta \in (0,1)$ is the discount factor, utility function $u\colon \mathbb{R}_+ \to [0, \bar{u}]$ is bounded, continuous, strictly increasing, strictly concave, and continuously differentiable, and satisfies $u(0) = 0$, $u'(0) = \infty$ and $\lim_{c\to\infty} u'(c) = 0$. At the beginning of each period, each agent receives an idiosyncratic random endowment $y \in Y = \{y_l, y_h\}$ with $0 < y_l < y_h$. The endowment y is i.i.d. across agents and across periods: the probability of $y = y_j$ is $\pi_j \in (0,1)$, $j = l, h$, and $\pi_l + \pi_h = 1$.

There is an intrinsically useless durable asset called money. The initial distribution of money among agents is $\mu_0 \in \Delta(\mathbb{R}_+)$. In each period after the endowment shock is realized, agents may trade their money holdings against the consumption good in a competitive market. There is no borrowing and lending. Agents can save only in the form of money.

At the end of each period after trading, the government levies a proportional tax on each individual's money balance at a constant rate $\tau \in [0,1]$. It redistributes the tax proceeds evenly among all agents (i.e., via lump-sum transfers). This implies that the aggregate stock of money is constant over time. Normalize the aggregate stock of money to be 1 per capita, and then the lump-sum transfer of money to each agent at the end of each period is τ.

1. (16 points) Given a government policy variable τ, define a stationary recursive competitive equilibrium (RCE). Such an equilibrium is *monetary* if money has value and is used in transactions.

2. (14 points) Consider an individual agent's recursive problem in a stationary monetary RCE. State a set of sufficient conditions that guarantee the existence and uniqueness of the value function and the additional conditions that are sufficient for the existence and uniqueness of the optimal policy function (i.e., a solution to the value function). Are these conditions satisfied?

3. (5 points) Consider an individual agent's problem in part 2. Rewrite the Bellman equation so that an individual state is represented by a single variable.

4. (10 points) Write down the Euler equation for the reformulated Bellman equation in part 3. Use it to show that a stationary monetary equilibrium does not exist if τ is too high.

Question 3. Lack of Commitment

Consider a small open economy where the domestic government has access to the technology given by $y = sf(k)$, where k is foreign investment, $s \in S = \{s_1, s_2, ..., s_N\}$ is an *iid* productivity shock with probabilities $\pi_s > 0$ where $\sum_s \pi_s = 1$, and $f(.)$ is a concave function. The domestic government receives foreign investment and borrows and lends in a one period bond. The utility of the domestic government is linear in consumption, $U(c) = c$ for $c \geq 0$. The domestic government discounts the future at rate $\beta \in (0, 1)$. The world interest rate is given by r and capital depreciates at a rate δ.

An efficient allocation can be expressed as the optimal contract between foreigners and the domestic government. In recursive form, the state variable is the promised utility to the domestic government, v. Denote by $B(v)$ the value function of the foreigners given the promised value v to the domestic government. The Bellman equation for the problem is:

$$B(v) = \max_{\{c(s),w(s),k\}} \sum_{s\in S} \pi(s)\Big(sf(k) - (r+\delta)k - c(s) + \frac{1}{1+r}B(w(s))\Big)$$

$$\text{subject to:} \quad c(s) \geq 0$$

$$v = \sum_{s\in S} \pi(s)\big(c(s) + \beta w(s)\big)$$

Assume **full commitment** from both the foreigner and the domestic government.

1. (3 points) Interpret the Bellman equation, including the objective, constraints, and choice variables.

2. (3 points) Obtain the first-order conditions of the problem.

3. (5 points) Solve for the optimal level of foreign direct investment, denoted by k^*.

For the rest of the question, suppose the domestic government lacks commitment (**one-sided lack of commitment**). If it deviates from the optimal contract when the shock is s and capital is k, it receives a payoff of $sf(k)$ in that period (i.e. all the output) and then zero thereafter. That means, the following participation constraints need to be added to the problem above:

$$c(s) + \beta w(s) \geq sf(k) \quad \forall s \in S$$

Ignore the non-negativity constraint of consumption. Assume that $B(v)$ is strictly concave and differentiable.

4. (3 points) Write down the Lagrangian and obtain the first order conditions of the problem without commitment.

5. (6 points) Show that either **all participation constraints are binding** (note there is one participation constraint per state s) or **all are relaxed** (i.e. the multipliers of the participation constraints are either all zero or all positive).

6. (3 points) What is the maximum value for the Lagrange multiplier on the promised keeping constraint?

7. (5 points) Using the first order conditions, show how does the capital k relate to k^* when all the participation constraints are relaxed (all the multipliers are equal to zero)? What about when all the participation constraints are binding (all the multipliers are positive)?

8. (3 points) Let $v^* = \sum_{s\in S} \pi(s)sf(k^*)$. Show that all the participation constraints are relaxed (all the multipliers are equal to zero) for any $v > v^*$.

9. (3 points) Show that if $v < v^*$, then all the participation constraints bind (all the multipliers are positive).

Assume that $\beta(1+r) < 1$.

10. (5 points) From the first order conditions of the problem show that for any v, $w(s) = \hat{v}$, where $\hat{v}$ solves $B'(\hat{v}) = -\beta(1+r)$.

11. (3 points) Suppose $v = \hat{v}$. Use the envelope condition together with the first order conditions to argue that $k(\hat{v})$ is given by the value $\bar{k}$ that solves

$$\sum_{s\in S} \pi(s)sf'(\bar{k}) = \frac{r+\delta}{\beta(1+r)}$$

12. (3 points) How does $\bar{k}$ vary with β? Compare with k^* in (3). Provide an economic intuition for this difference.

Question 4. Limitations of Hall's (1978) Random Walk Hypothesis

Consider a representative consumer who chooses consumption $\{C_t\}_{t=0}^{\infty}$ and assets $\{A_{t+1}\}_{t=0}^{\infty}$ to maximize the expected value of lifetime utility:

$$\mathbb{E}_0 \left\{ \sum_{t=0}^{\infty} \beta^t \left[(C_t + q_t) - \frac{(C_t + q_t)^2}{2} \right] \right\},$$

subject to the sequential budget constraint:

$$C_t + \frac{A_{t+1}}{1+r} = Y_t + A_t, \quad \forall t \geq 0$$

and the no-Ponzi-Scheme condition, where A_0 is given. The income process $\{Y_t\}_{t=0}^{\infty}$ is stochastic. The sequence $\{q_t\}_{t=0}^{\infty}$ is deterministic, where q_t represents the quality of consumption goods in period t. Assume that C_t is always in the range where $u'(C_t)$ is positive. For simplicity, assume that $(1+r)\beta = 1$. (Note: the consumer's problem is the same as in Hall (1978) if $q_t = 0$ for all $t \geq 0$.)

1. **(10 points)** Derive the Euler equation relating C_t and $\mathbb{E}_t(C_{t+1})$.

2. **(10 points)** Does consumption $\{C_t\}_{t=0}^{\infty}$ follow a random walk in general? What additional condition is required for Hall's random walk hypothesis to hold? Show your derivation and discuss the economic intuition for this condition.

3. **(5 points)** Suppose one can observe quality-adjusted consumption, $Q_t = C_t + q_t$, in the data. Does the sequence $\{Q_t\}_{t=0}^{\infty}$ follow a random walk? Show your derivation.

4. **(10 points)** Suppose, however, the data only provide total quality-adjusted consumption over two-period intervals: $\{Q_0 + Q_1, Q_2 + Q_3, Q_4 + Q_5 \cdots\}$. That is, one can only observe $\{\widetilde{Q}_s\}_{s=0}^{\infty}$, where

$$\widetilde{Q}_s = Q_{2s} + Q_{2s+1},$$

for each $s = 0, 1, 2, \cdots$ in the data. Does the observed quality-adjusted consumption $\{\widetilde{Q}_s\}_{s=0}^{\infty}$ follow a random walk? Compare your answer to part 3 and discuss.

5. **(10 points)** Suppose, instead, the data does not provide the total amount over each two-period interval, but the amount in the second of the two periods: $\{Q_1, Q_3, Q_5, ...\}$. That is, one can only observe $\{\widehat{Q}_s\}_{s=0}^{\infty}$, where

$$\widehat{Q}_s = Q_{2s+1},$$

for each $s = 0, 1, 2, \cdots$ in the data. Does the observed quality-adjusted consumption $\{\widehat{Q}_s\}_{s=0}^{\infty}$ follow a random walk? Compare your answer to part 4 and discuss.