Comprehensive exam 2025 (Econ 501, Pinkse)

Random variables are typeset in bold.

1. (Asymptotic theory)

   (a) Let $\hat{\boldsymbol{\sigma}}^2, \sigma_0^2$ denote the sample and population variance, respectively, of an i.i.d. sample of size n with finite fourth moments. Derive the limit distribution of $\sqrt{n}(\hat{\boldsymbol{\sigma}}^2 - \sigma_0^2)$.

   (b) Establish the convergence rate of $\bar{\mathbf{x}}$ if $\bar{\mathbf{x}}$ is the sample mean of an i.i.d. sequence of random variables with $0 < \mathbb{E}\|\mathbf{x}\|^p < \infty$ for some $1 < p < \infty$. Your answer should be a function of p and work for all such values of p.

   (c) Let $\hat{\boldsymbol{\sigma}}^2$ be the sample variance of an i.i.d. sample (of size n) of unit variance random variables with finite fourth moments. Let $\mathbf{z}_n$ be a χ_n^2-distributed random variable independent of $\hat{\boldsymbol{\sigma}}^2$. Derive $\limsup_{n\to\infty} \mathbb{P}(n\hat{\boldsymbol{\sigma}}^2 > \mathbf{z}_n)$.

2. Suppose that $\mathbf{x}, \mathbf{y}$ are jointly normal with mean zero, unit variance, and correlation ρ for which $\rho^2 = 3/4$. Compute $\mathbb{E}(\mathbf{y} \mid \mathbf{y} \le \mathbf{x}, \mathbf{x} = 0)$.

3. Let $\mathbf{x}_n$ for any n be a continuously distributed random variables with bounded support and density $f_n \le 1$. Suppose further that $\mathbf{x}_n \xrightarrow{p} 0$. Prove or disprove the statement "$\mathbb{E}\mathbf{x}_n$ necessarily converges to zero."

4. Suppose that for any i, $\mathbf{x}_i$ is such that $\mathbb{E}\mathbf{x}_i = 3$ and $\mathbb{P}(\mathbf{x}_i \ge 1) = 1$. Show that $\mathbb{P}(\sum_{i=1}^n \log \mathbf{x}_i > 3n) \le 1/3$. *You may **not** assume anything beyond the conditions stated.*

5. Let $\{\mathbf{u}_i\}$ be i.i.d. standard exponential (i.e. $\mathbf{u}_i$ has density function $\exp(-u)\mathbb{1}(u \ge 0)$.).

   (a) Determine the cumulants of $\mathbf{u}_i$.

   (b) Suppose further that $\mathbf{y}_i = \theta_0 \mathbf{x}_i \mathbf{u}_i$ where $\mathbf{x}_i$ has bounded support on the positive halfline and $\theta_0 > 0$. Determine the (conditional) maximum likelihood estimator $\hat{\boldsymbol{\theta}}$ of θ_0 if the $\mathbf{u}_i$'s are independent of the $\mathbf{x}_i$'s.

6. Consider the probability space $(\Omega, \mathcal{B}, \lambda)$ with $\Omega = [0, 1]$, $\mathcal{B}$ the Borel field, and λ the Lebesgue measure. Consider the function $\mathbf{x} : \Omega \to \mathbb{R}$ with

$$\mathbf{x}(\omega) = \begin{cases} \sqrt{\omega}, & \text{if } \omega \text{ is rational,} \\ 1 - \omega^2, & \text{if } \omega \text{ is irrational.} \end{cases}$$

   Is $\mathbf{x}(\omega)$ a random variable? Show that it is(n't).

Comprehensive exam (Part Econ 510, Guggenberger)

1. a) Explain precisely what is meant by i) point identification, ii) partial identification, and iii) weak identification. Which concept implies which (if any)? Provide meaningful examples.

b) Define precisely what is meant for a test to i) control the limiting null rejection probability, ii) have correct size, and iii) have correct asymptotic size. Provide an example of empirical interest where a test controls the limiting null rejection probability but whose asymptotic size exceeds the nominal size.

2. Under Assumptions EE3 and CF-NS (stated below) we derived the limiting distribution of the GMM estimator $\widehat{\theta}_n$ with nonsmooth stochastic criterion function, namely

$$\sqrt{n}(\widehat{\theta}_n - \theta_0) \to_d N(0, (\Gamma'\Gamma)^{-1}\Gamma' V_0 \Gamma (\Gamma'\Gamma)^{-1}).$$

a) Provide estimators for V_0 and Γ and discuss their consistency under appropriate conditions.

Consider the example of Quantile Regression, $(Y_i, X_i')'$ i.i.d.

$$Y_i = X_i'\theta_0 + U_i,$$

where the τ-quantile of U_i conditional on X_i equals 0 for some $\tau \in (0,1)$.

b) Derive what V_0 and Γ are in this case.

c) Find a simple expression for $(\Gamma'\Gamma)^{-1}\Gamma' V_0 \Gamma (\Gamma'\Gamma)^{-1}$ in this case.

Setup and Assumptions for the GMM case with nonsmooth stochastic criterion function:

Let $Q_n(\theta) = ||\overline{g}_n(\theta)||$, where $\overline{g}_n(\theta) = n^{-1}\sum_{i=1}^n g(W_i, \theta)$, and $g(\theta) = Eg(W_i, \theta)$.

Assumption CF-NS: (i) θ_0 *is in the interior of* Θ.

(ii) $g(\theta)$ *is differentiable at* θ_0 *with* $\Gamma = (\partial/\partial\theta')g(\theta_0)$ of full rank $d \leq k$.

(iii) $g(\theta_0) = 0$.

(iii) $\sqrt{n}\overline{g}_n(\theta_0) \to_d N(0, V_0)$.

(iv) For every sequence of positive constants $\{\delta_n\}_{n\geq 1}$ that converges to zero,

$$\sup_{\theta\in\Theta, ||\theta-\theta_0||<\delta_n} \sqrt{n}\, ||\overline{g}_n(\theta) - g(\theta) - \overline{g}_n(\theta_0)|| \to_p 0.$$

Assumption EE3: (i) $Q_n(\widehat{\theta}_n) = \inf_{\theta\in\Theta} Q_n(\theta) + o_p(n^{-1/2})$ and (ii) $\widehat{\theta}_n \to_p \theta_0$.

3. Suppose that $X_i \sim N(\mu_i, 1)$, $i = 1, ..., n$ are independent. Consider the following five tests of the null hypothesis $H_0 : \mu_1 = ... = \mu_n = 0$ versus the alternative H_1 : "not H_0" with acceptance regions

$$\begin{aligned} A_1 &= \{(x_1, ..., x_n) : \textstyle\sum_{i=1}^n x_i \leq k_1\}, \\ A_2 &= \{(x_1, ..., x_n) : |\textstyle\sum_{i=1}^n x_i| \leq k_2\}, \\ A_3 &= \{(x_1, ..., x_n) : x_i^2 \leq k_3 \text{ for all } i = 1, ..., n\}, \\ A_4 &= \{(x_1, ..., x_n) : x_1^2 \leq k_4\}, \\ A_5 &= \{(x_1, ..., x_n) : \textstyle\sum_{i=1}^n x_i^2 \leq k_5\}, \end{aligned}$$

where k_j, $j = 1, ..., 5$ are constants (that possibly depend on the sample size n) chosen to ensure that the tests have size equal to α for some $\alpha \in (0, 1)$.

a) Determine k_j, $j = 1, ..., 5$.

b) Let $n = 2$ from now on. Determine the power functions of the first and the fourth test.

c) Show that none of the five tests is (weakly) more powerful than all the other tests considered here uniformly over the alternative space $\{(\mu_1, \mu_2) : -\infty < \mu_1, \mu_2 < \infty\}$.

4. Consider the relation $y = X\theta^* + w \in R^n$ with $X \in R^{n \times p}$ where the objetive is estimation of θ^* after observing the data (y, X). Possibly, p is much larger than n.

a) Define the LASSO estimator. What do we mean by the "oracle property" of LASSO?

b) Present a set of high level assumptions for given θ^*, the penalty term λ_n used in the implementation of LASSO, X, and w such that the oracle property is satisfied. (A formal presentation is best but an intuitive verbal description is sufficient). Critically discuss these assumptions.

c) You did finite sample simulations for the LASSO estimator. Describe the performance of LASSO relative to the Ridge estimator under various choices of the DGP.

d) Show that the Ridge estimator can be calculated via simple OLS regression when augmenting y and X appropriately. Namely, replace y by $(y', 0_p')'$ and X by $\binom{X}{\sqrt{\lambda_n} I_p}$.

5. In the Heckit model (Heckman, 1979), using the same notation/assumptions as in class ($y^* = x'\theta + \varepsilon$, $y = dy^*$, $d = 1(x'\pi_1 + z'\pi_2 \geq -\eta$, joint normality of (ε, η)...)

a) show that

$$E(y|x, z, d = 1) = x'\theta + \sigma_{\varepsilon\eta}\lambda(x'\pi_1 + z'\pi_2),$$

where $\lambda(s) = \phi(s)/\Phi(s)$.

b) Identify π_1 and π_2 from a probit regression of d onto x and z.

c) Specify conditions under which one can identify θ and $\sigma_{\varepsilon\eta}$.